Bousfield-Friedlander theorem

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1 Proper Model Category

In a model category, fibrations are preserved under pullbacks, and cofibrations are preserved under pushouts. But weak equivalences, in general does not have such closure property. In a *proper model category*, weak equivalences are preserved under certain pullbacks and pushouts.

Definition 1.1. (Left proper model category) A model category is called *left proper* if weak equivalences are preserved under pushouts along cofibrations, i.e, for every $f : B \to X$ in $we(\mathcal{C})$ and for every $i : B \to A$ in $cof(\mathcal{C})$, the pushout morphism $i_*f : A \to A \coprod_B X$ is in $we(\mathcal{C})$.

$$\begin{tikzcd} A \coprod_B X & X \arrow[l] \\ A \arrow[u, "i_*f"] & B \arrow[l, "i"] \arrow[u, "f"'] \end{tikzcd}$$

Definition 1.2. (Right proper model category) A model category is called *right proper* if weak equivalences are preserved under pullbacks along fibrations, i.e, for every $f : A \to Y$ in $we(\mathcal{C})$ and for every $p : X \to Y$ in $fib(\mathcal{C})$, the pullback morphism $p^*f : X \times_Y A \to X$ is in $we(\mathcal{C})$.

$$\begin{tikzcd} X \times_Y A \arrow[r] \arrow[d, "p^*f"'] & A \arrow[d, "f"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

Definition 1.3. (Proper model category) A model category is called *proper* if it is both *left proper* and *right proper*.

2 Homotopy fiber squares

Definition 2.1. Let $\mathcal{C}$ be a proper model category. A commutative square in $\mathcal{C}$,

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
B \arrow[r, "g"'] & D
\end{tikzcd}$$

is called a *homotopy fiber square* if for some factorization $C \xrightarrow{i_f} E(f) \xrightarrow{p_f} D$ of f where i_f is in $we(\mathcal{C})$ and p_f is in $fib(\mathcal{C})$,

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & C \arrow[dr, "i_f"] \arrow[dd] & \\
& B \times_D E(f) \arrow[rr] \arrow[dl] & & E(f) \arrow[dl, "p_f"] \\
B \arrow[rr, "g"'] & & D &
\end{tikzcd}$$

the induced morphism $A \to B \times_D E(f)$ is in $we(\mathcal{C})$.

Proposition 2.1. Let $\mathcal{C}$ be a proper model category. Then the following statements are equivalent:-

1. The commutative square in $\mathcal{C}$,

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
B \arrow[r, "g"'] & D
\end{tikzcd}$$

   is a homotopy fiber square.

2. For some factorization $B \xrightarrow{i_g} E(g) \xrightarrow{p_g} D$ of g where i_g is in $we(\mathcal{C})$ and p_g is in $fib(\mathcal{C})$,

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & C \arrow[dd, "f"] \\
& E(g) \times_D C \arrow[ur] \arrow[dd] & \\
B \arrow[rr] \arrow[dr, "i_g"'] & & D \\
& E(g) \arrow[ur, "p_g"'] &
\end{tikzcd}$$

   the induced morphism $A \to E(g) \times_D C$ is in $we(\mathcal{C})$.

Proof. ($1 \implies 2$) Suppose the commutative square in (1) is a homotopy fiber square and take any factorization $B \xrightarrow{i_g} E(g) \xrightarrow{p_g} D$ of g where i_g is in $we(\mathcal{C})$ and

p_g is in $fib(\mathcal{C})$. Then consider the commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "\hat{i_g}"] \arrow[d, "\hat{i_f}"'] & E(g)\times_D C \arrow[r, "\hat{p_g}"] \arrow[d, "\tilde{i_f}"] & C \arrow[d, "i_f"] \\
B\times_D E(f) \arrow[r, "\tilde{i_g}"] \arrow[d, "\hat{p_f}"'] & E(g)\times_D E(f) \arrow[r, "\tilde{p_g}"] \arrow[d, "\tilde{p_f}"] & E(f) \arrow[d, "p_f"] \\
B \arrow[r, "i_g"'] & E(g) \arrow[r, "p_g"'] & D
\end{tikzcd}$$

where $C \xrightarrow{i_f} E(f) \xrightarrow{p_f} D$ is one such factorization of f, i_f in $we(\mathcal{C})$ and p_f in $fib(\mathcal{C})$, such that $\hat{i_f}$ is in $we(\mathcal{C})$. Now the bottom right square being a pullback, $\tilde{p_g}$ and $\tilde{p_f}$ is in $fib(\mathcal{C})$. Again $\mathcal{C}$ being proper, and the bottom left and top right squares being pullbacks, $\tilde{i_f}$, $\tilde{i_g}$ is in $we(\mathcal{C})$. Finally by the 2-out-of-3 property, $\hat{i_g}$ is in $we(\mathcal{C})$.
($2 \implies 1$) Similar argument as above, just in this case we assume that $\hat{i_g}$ is in $we(\mathcal{C})$, and conclude that $\hat{i_f}$ is in $we(\mathcal{C})$. □

Therefore, we conclude that (1) and (2) holds for every such factorization of f and g.

Proposition 2.2. Let $\mathcal{C}$ be a proper model category, and consider the commutative square.

$$\begin{tikzcd}
A \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{f}"'] & C \arrow[d, "g"] \\
B \arrow[r, "f"'] & D
\end{tikzcd}$$

If f is in $we(\mathcal{C})$, then $\tilde{f}$ is in $we(\mathcal{C})$ iff the above square is a homotopy fiber square.

Proof. Let $g = p_g i_g$ be a factorization of p, where i_g is in $we(\mathcal{C})$ and p_g is in $fib(\mathcal{C})$. Consider the commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{i}"'] & C \arrow[d, "i_g"] \\
B\times_D E(g) \arrow[r, "p_g^* f"] \arrow[d, "f^* p_g"'] & E(g) \arrow[d, "p_g"] \\
B \arrow[r, "f"'] & D
\end{tikzcd}$$

where $p_g^* f$ is in $we(\mathcal{C})$, since $\mathcal{C}$ is proper. Now, if $\tilde{f}$ is in $we(\mathcal{C})$, then by the 2-out-of-3 property, $\tilde{i}$ is in $we(\mathcal{C})$. Conversely, if $\tilde{i}$ is in $we(\mathcal{C})$, then again by the 2-out-of-3 property, $\tilde{f}$ is in $we(\mathcal{C})$.

□

Proposition 2.3. Let $\mathcal{C}$ be a proper model category and consider the commutative square,

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & E \arrow[d] \\ B \arrow[r] & D \arrow[r] & F \end{tikzcd}$$

where the right square is a homotopy fiber square. Then the left square is also a homotopy fiber square iff the total rectangle is a homotopy fiber square.

Proof. Consider the commutative diagram,

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dd, "h"'] \arrow[dr, "i_h"] & & C \arrow[rr, "\beta"] \arrow[dd] \arrow[dr, "i_g"] & & E \arrow[dd] \arrow[dr, "i_f"] & \\ & R \arrow[rr, "g"] \arrow[dl, "p_h"] & & Q \arrow[rr, "f"] \arrow[dl, "p_g"] & & P \arrow[dl, "p_f"] \\ B \arrow[rr, "\gamma"'] & & D \arrow[rr, "\delta"'] & & F & \end{tikzcd}$$

where we first factorize $f = p_f i_f$, where i_f is in $we(\mathcal{C})$, and p_f is in $fib(\mathcal{C})$. Then take the pullback of p_f along δ in order to factorize $g = p_g i_g$. Since p_f is in $fib(\mathcal{C})$, p_g is in $fib(\mathcal{C})$. Also, since the right square is a homotopy fiber square, i_g is in $we(\mathcal{C})$. Again take the pullback of p_g along γ, to factorize $h = p_h i_h$. Now suppose the left square is a homotopy fiber square. Then i_h is in $we(\mathcal{C})$. But p_h is a pullback of p_f along $\delta\gamma$, which implies that the total rectangle is a homotopy fiber square.
Conversely, let the total rectangle be a homotopy fiber square. Then i_h is in $we(\mathcal{C})$, which implies that that the left square in a homotopy fiber square.

□

Proposition 2.4. Every retract of a homotopy fiber square in $\mathcal{C}^{\square}$ is a homotopy fiber square.

3 $\mathcal{Q}$-structures on proper model categories

Let $\mathcal{C}$ be a proper model category and $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ be an endofunctor. A morphism $f : X \to Y$ in $\mathcal{C}$ is called

- a *$\mathcal{Q}$-equivalence* if $\mathcal{Q}(f) : \mathcal{Q}(X) \to \mathcal{Q}(Y)$ is a weak equivalence in $\mathcal{C}$.
- a *$\mathcal{Q}$-cofibration* if f is a cofibration in $\mathcal{C}$.
- a *$\mathcal{Q}$-fibration* if f has the *right lifting property* with respect to $\mathcal{Q}$-trivial cofibrations.

Definition 3.1. (Quillen idempotent monad) Let $\mathcal{C}$ be a proper model category. A *Quillen idempotent monad* on $\mathcal{C}$ is

- an endofunctor $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$

- a natural transformation $\eta : \mathbf{1}_{\mathcal{C}} \to \mathcal{Q}$

such that

1. $\mathcal{Q}$ is homotopical, i.e., $\mathcal{Q}$ preserves weak equivalences.
2. For every object X in $\mathcal{C}$, the morphisms, $\mathcal{Q}(\eta_X), \eta_{\mathcal{Q}(X)} : \mathcal{Q}(X) \to \mathcal{Q}(\mathcal{Q}(X))$ are weak equivalences.
3. For a pullback square in $\mathcal{C}$,

$$\begin{tikzcd} X \times_Y A \arrow[r] \arrow[d, "p^*f"'] & A \arrow[d, "f"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

   if p is a $\mathcal{Q}$-fibration and f is a $\mathcal{Q}$-equivalence, then p^*f is a $\mathcal{Q}$-equivalence.

4. For a pushout square in $\mathcal{C}$,

$$\begin{tikzcd} A \coprod_B X & X \arrow[l] \\ A \arrow[u, "i_*f"] & B \arrow[l, "i"] \arrow[u, "f"'] \end{tikzcd}$$

   if i is a $\mathcal{Q}$-cofibration and f is a $\mathcal{Q}$-equivalence, then i_*f is a $\mathcal{Q}$-equivalence.

For a Quillen idempotent monad $\mathcal{Q}$ on $\mathcal{C}$, let $\mathcal{C}^{\mathcal{Q}}$ denote the category $\mathcal{C}$ equipped with $\mathcal{Q}$-equivalences, $\mathcal{Q}$-fibrations and $\mathcal{Q}$-cofibrations.

Lemma 3.1. *A morphism $f : X \to Y$ in $\mathcal{C}$ is in $we(\mathcal{C}) \cap fib(\mathcal{C})$ iff f is a $\mathcal{Q}$-trivial fibration.*

Proof. Let $f : X \to Y$ be in $we(\mathcal{C}) \cap fib(\mathcal{C})$. Then by (1), f is a $\mathcal{Q}$-equivalence. Now consider the commutative square,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where i is a $\mathcal{Q}$-trivial cofibration. Since i is a $\mathcal{Q}$-cofibration, i is in $cof(\mathcal{C})$ and f being in $we(\mathcal{C}) \cap fib(\mathcal{C})$, f has the right lifting property with respect to i. Hence, f is a $\mathcal{Q}$-trivial fibration.

Conversely, let $f : X \to Y$ in $\mathcal{C}$ be a $\mathcal{Q}$-trivial fibration. Then $f = p_f i_f$ where i_f is in $cof(\mathcal{C})$ and p_f is in $we(\mathcal{C}) \cap fib(\mathcal{C})$. Now $\mathcal{Q}(f) = \mathcal{Q}(p_f)\mathcal{Q}(i_f)$, where $\mathcal{Q}(f)$ is in $we(\mathcal{C})$. Also by (1), $\mathcal{Q}(p_f)$ is in $we(\mathcal{C})$. By the 2-out-of-3 property, i_f is a $\mathcal{Q}$-equivalence. So, i_f is a $\mathcal{Q}$-trivial cofibration, hence has the left lifting property with respect to f. By the retract argument, f is a retract of p_f, therefore, f is in $we(\mathcal{C}) \cap fib(\mathcal{C})$. $\square$

Lemma 3.2. *If a morphism $f : X \to Y$ is in $fib(\mathcal{C})$, and $\eta_X : X \to \mathcal{Q}(X)$, $\eta_Y : Y \to \mathcal{Q}(Y)$ are in $we(\mathcal{C})$, then f is a $\mathcal{Q}$-fibration.*

Proof. For any commutative square,

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a $\mathcal{Q}$-trivial cofibration, it suffices to show that there exists a lift. First, we factorize the functorial image of the above commutative square,

$$\begin{tikzcd} \mathcal{Q}(A) \arrow[r, "j_\alpha"] \arrow[d, "\mathcal{Q}(i)"'] & Z \arrow[r, "\rho_\alpha"] \arrow[d, "\pi"] & \mathcal{Q}(X) \arrow[d, "\mathcal{Q}(f)"] \\ \mathcal{Q}(B) \arrow[r, "j_\beta"] & W \arrow[r, "\rho_\beta"] & \mathcal{Q}(Y) \end{tikzcd}$$

where, j_α, j_β are in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and ρ_α, ρ_β are in $fib(\mathcal{C})$. We obtain this factorization in the following way: first factorize $\mathcal{Q}(\beta) = \rho_\beta j_\beta$, where j_β is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and ρ_β is in $fib(\mathcal{C})$. Then take the pullback $(\mathcal{Q}(f))^*(\rho_\beta) : W \times_{\mathcal{Q}(Y)} \mathcal{Q}(X) \to \mathcal{Q}(X)$ of ρ_β along $\mathcal{Q}(f)$, which is in $fib(\mathcal{C})$, since ρ_β is. By the universal property, there exists an unique morphism $h : \mathcal{Q}(A) \to W \times_{\mathcal{Q}(Y)} \mathcal{Q}(X)$, such that $\mathcal{Q}(\alpha) = (\mathcal{Q}(f))^*(\rho_\beta)h$. Again factorize $h = qj_\alpha$, where, j_α is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and q is in $fib(\mathcal{C})$. Finally, set $\rho_\alpha = (\mathcal{Q}(f))^*(\rho_\beta)q$ and $\pi = (\rho_\beta)^*(\mathcal{Q}(f))q$.

$$\begin{tikzcd} Z \arrow[drr, "q"] \arrow[dddrr, "\pi"] & \mathcal{Q}(A) \arrow[l, "j_\alpha"'] \arrow[rr, "\mathcal{Q}(\alpha)"] \arrow[dr, dashed, "!h"] \arrow[dd, "\mathcal{Q}(i)"] & & \mathcal{Q}(X) \arrow[dd, "\mathcal{Q}(f)"] \\ & & W \times_{\mathcal{Q}(Y)} \mathcal{Q}(X) \arrow[ur, "\mathcal{Q}(f)^*\rho_\beta"] \arrow[dd] & \\ & \mathcal{Q}(B) \arrow[rr, "\mathcal{Q}(\beta)"] \arrow[dr, "j_\beta"'] & & \mathcal{Q}(Y) \\ & & W \arrow[ur, "\rho_\beta"'] & \end{tikzcd}$$

Now consider the pullback of the right square along the η-naturality square on f,

$$
\begin{tikzcd}
 & & & Z \times_{\mathcal{Q}(X)} X \arrow[d, "{(\pi,f)}"] \arrow[dr] \arrow[ddl] & \\
A \arrow[urrr, "{(j_\alpha \eta_A, \alpha)}"] \arrow[rrrr, dotted] \arrow[d, "i"'] \arrow[dr, "\eta_A"'] & & & W \times_{\mathcal{Q}(Y)} Y \arrow[dr] \arrow[ddl] & X \arrow[d, "f"] \arrow[dl, "\eta_X"'] \\
B \arrow[urrr, "{(j_\beta \eta_B, \beta)}"] \arrow[rrrr, dotted] \arrow[dr, "\eta_B"'] & \mathcal{Q}(A) \arrow[r, "j_\alpha"] \arrow[d, "\mathcal{Q}(i)"'] & Z \arrow[r, "\rho_\alpha"] \arrow[d, "\pi"] & \mathcal{Q}(X) \arrow[d, "\mathcal{Q}(f)"] & Y \arrow[dl, "\eta_Y"] \\
 & \mathcal{Q}(B) \arrow[r, "j_\beta"] & W \arrow[r, "\rho_\beta"] & \mathcal{Q}(Y) &
\end{tikzcd}
$$

we obtain the commutative diagram,

$$
\begin{tikzcd}
A \arrow[r, "{(j_\alpha \eta_A, \alpha)}"] \arrow[d, "i"'] & Z \times_{\mathcal{Q}(X)} X \arrow[r] \arrow[d, "{(\pi,f)}"] & X \arrow[d, "f"] \\
B \arrow[r, "{(j_\beta \eta_B, \beta)}"] & W \times_{\mathcal{Q}(Y)} Y \arrow[r] & Y
\end{tikzcd}
$$

where the left square is due to the universal property of the pullback. Now we show that (π, f) is in $we(\mathcal{C})$. For that, consider the diagram,

$$
\begin{tikzcd}
\mathcal{Q}(A) \arrow[r, "j_\alpha"] \arrow[d, "\mathcal{Q}(i)"'] & Z \arrow[d, "\pi"] & Z \times_{\mathcal{Q}(X)} X \arrow[l, "(\rho_\alpha)^* \eta_X"'] \arrow[d, "{(\pi,f)}"] \\
\mathcal{Q}(B) \arrow[r, "j_\beta"'] & W & W \times_{\mathcal{Q}(Y)} Y \arrow[l, "(\rho_\beta)^* \eta_Y"]
\end{tikzcd}
$$

Since $\mathcal{C}$ is proper, $(\rho_\alpha)^*\eta_X$ and $(\rho_\beta)^*\eta_Y$ are in $we(\mathcal{C})$. By the 2-out-of-3 property, π is in $we(\mathcal{C})$, and which further implies that (π, f) is in $we(\mathcal{C})$. Finally factorize $(\pi, f) = rk$, where k is in $cof(\mathcal{C})$ and r is in $we(\mathcal{C}) \cap fib(\mathcal{C})$. By the 2-out-of-3 property, k is in $we(\mathcal{C}) \cap cof(\mathcal{C})$. Then in the following commutative diagram,

$$
\begin{tikzcd}
A \arrow[r, "{(j_\alpha \eta_A, \alpha)}"] \arrow[d, "\mathbf{1}_A"'] & Z \times_{\mathcal{Q}(X)} X \arrow[r] \arrow[d, "k"] & X \arrow[d, "f"] \\
A \arrow[r] \arrow[d, "i"'] & \bigstar \arrow[r] \arrow[d, "r"] \arrow[ur, dashed, "l"] & Y \arrow[d, "\mathbf{1}_Y"] \\
B \arrow[r, "{(j_\beta \eta_B, \beta)}"'] \arrow[ur, dashed, "l'"] & W \times_{\mathcal{Q}(Y)} Y \arrow[r] & Y
\end{tikzcd}
$$

i being in $cof(\mathcal{C})$ and r in $we(\mathcal{C}) \cap fib(\mathcal{C})$, l' exists. Similarly, k being in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and f in $fib(\mathcal{C})$, l exists. Hence, ll' is our desired lift.

$\square$

Theorem 3.3. *(Bousfield-Friedlander theorem) $\mathcal{C}^{\mathcal{Q}}$ is a proper model catgeory, where $we(\mathcal{C}^{\mathcal{Q}})$, $cof(\mathcal{C}^{\mathcal{Q}})$ and $fib(\mathcal{C}^{\mathcal{Q}})$ are $\mathcal{Q}$-equivalences, $\mathcal{Q}$-fibrations and $\mathcal{Q}$-cofibrations respectively.*

Proof. Since $\mathcal{C}$ is a model category, $\mathcal{C}^{\mathcal{Q}}$ has limits and colimits. Suppose $h = gf$, and two of the three morphisms f, g and h are in $we(\mathcal{C}^{\mathcal{Q}})$. Then $\mathcal{Q}(h) = \mathcal{Q}(g)\mathcal{Q}(f)$, and by the 2-out-of-3 property of $\mathcal{C}$, the third morphism is also in $we(\mathcal{C}^{\mathcal{Q}})$. This proves the 2-out-of-3 property of $\mathcal{C}^{\mathcal{Q}}$. Now, since $cof(\mathcal{C}^{\mathcal{Q}}) = cof(\mathcal{C})$, and by Lemma 2.1, $we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}}) = we(\mathcal{C}) \cap fib(\mathcal{C})$, we have that $(cof(\mathcal{C}^{\mathcal{Q}}), we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}}))$ is a weak factorization system. On the other hand, by the definition of $fib(\mathcal{C}^{\mathcal{Q}})$, we have $fib(\mathcal{C}^{\mathcal{Q}}) = (we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}}))^{\boxslash}$.
Now we consider a morphism $f : X \to Y$ in $\mathcal{C}^{\mathcal{Q}}$. Then we factorize $\mathcal{Q}(f)$,

$$\mathcal{Q}(X) \xrightarrow{i} Z \xrightarrow{p} \mathcal{Q}(Y)$$

i in $we(\mathcal{C}) \cap cof(\mathcal{C})$, hence in $we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}})$ and p in $fib(\mathcal{C})$. In the η-naturality square,

$$\begin{tikzcd}
\mathcal{Q}(X) \arrow[r, "i"] \arrow[d, "\eta_{\mathcal{Q}(X)}"'] & Z \arrow[r, "p"] \arrow[d, "\eta_Z"] & \mathcal{Q}(Y) \arrow[d, "\eta_{\mathcal{Q}(Y)}"] \\
\mathcal{Q}(\mathcal{Q}(X)) \arrow[r, "\mathcal{Q}(i)"'] & \mathcal{Q}(Z) \arrow[r, "\mathcal{Q}(p)"'] & \mathcal{Q}(\mathcal{Q}(Y))
\end{tikzcd}$$

since $\eta_{\mathcal{Q}(X)}, i$ and $\mathcal{Q}(i)$ are in $we(\mathcal{C})$, by 2-out-of-3 property, η_Z is in $we(\mathcal{C})$. So by Lemma 2.2, p is in $fib(\mathcal{C}^{\mathcal{Q}})$. Now, we factorize the η-naturality square on f, as the pullback corner morphism $\tilde{i}$ followed by the pullback $\tilde{p}$ of p along η_Y,

$$\begin{tikzcd}
X \arrow[r, "\tilde{i}"] \arrow[d, "\eta_X"'] & Z \times_{\mathcal{Q}(Y)} Y \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{\eta}"] & Y \arrow[d, "\eta_Y"] \\
\mathcal{Q}(X) \arrow[r, "i"] & Z \arrow[r, "p"] & \mathcal{Q}(Y)
\end{tikzcd}$$

By (3), $\tilde{\eta}$ is in $we(\mathcal{C}^{\mathcal{Q}})$, since η_Y is in $we(\mathcal{C}^{\mathcal{Q}})$ and p is in $fib(\mathcal{C}^{\mathcal{Q}})$. By the 2-out-of-3 property, $\tilde{i}$ is in $we(\mathcal{C}^{\mathcal{Q}})$. Also, since p is in $fib(\mathcal{C}^{\mathcal{Q}})$ and $fib(\mathcal{C}^{\mathcal{Q}}) = (we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}}))^{\boxslash}$, $\tilde{p}$ is in $fib(\mathcal{C}^{\mathcal{Q}})$. Finally, factorize $\tilde{i} = \tilde{q}\tilde{j}$, where $\tilde{j}$ is in $cof(\mathcal{C}^{\mathcal{Q}})$ and $\tilde{q}$ is in $we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}})$. By the 2-out-of-3 property of $\mathcal{C}^{\mathcal{Q}}$, $\tilde{j}$ is in $we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}})$ and $\tilde{p}\tilde{q}$ is our required morphism in $fib(\mathcal{C}^{\mathcal{Q}})$. Let j be a morphism in ${}^{\boxslash}fib(\mathcal{C}^{\mathcal{Q}})$, and factorize $j = rk$, k in $we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}})$ and r in $fib(\mathcal{C}^{\mathcal{Q}})$. By the retract argument, j is a retract of k. But, $we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}})$ is closed under retracts, since $cof(\mathcal{C}^{\mathcal{Q}}) = cof(\mathcal{C})$ and $we(\mathcal{C}) \subseteq we(\mathcal{C}^{\mathcal{Q}})$ are closed under retracts. Therefore, $we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}}) = {}^{\boxslash}fib(\mathcal{C}^{\mathcal{Q}})$ and $(we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}}), fib(\mathcal{C}^{\mathcal{Q}}))$ is a weak factorization system. The properness of $\mathcal{C}^{\mathcal{Q}}$ follows from (3) and (4). □

Proposition 3.4. A morphism $f : X \to Y$ in $\mathcal{C}$ is a $\mathcal{Q}$-fibration iff f is in $fib(\mathcal{C})$

and the η-naturality square on f,

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathcal{Q}(X) \arrow[d, "\mathcal{Q}(f)"] \\ Y \arrow[r, "\eta_Y"'] & \mathcal{Q}(Y) \end{tikzcd}$$

is a homotopy fiber square in $\mathcal{C}$.

Proof. Let $f : X \to Y$ in $\mathcal{C}$ be a $\mathcal{Q}$-fibration. Then by the definition of a $\mathcal{Q}$-fibration, f is in $fib(\mathcal{C})$. We factorize $\mathcal{Q}(f) = pi$, where i is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and p is in $fib(\mathcal{C})$, and then consider the commutative diagram,

$$\begin{tikzcd} X \arrow[r, "\tilde{i}"] \arrow[d, "\eta_X"'] & Z \times_{\mathcal{Q}(Y)} Y \arrow[r, "\tilde{p}"] \arrow[d, "p^*\eta_Y"] & Y \arrow[d, "\eta_Y"] \\ \mathcal{Q}(X) \arrow[r, "i"] \arrow[d, "\eta_{\mathcal{Q}(X)}"'] & Z \arrow[r, "p"] \arrow[d, "\eta_Z"] & \mathcal{Q}(Y) \arrow[d, "\eta_{\mathcal{Q}(Y)}"] \\ \mathcal{Q}(\mathcal{Q}(X)) \arrow[r, "\mathcal{Q}(i)"'] & \mathcal{Q}(Z) \arrow[r, "\mathcal{Q}(p)"'] & \mathcal{Q}(\mathcal{Q}(Y)) \end{tikzcd}$$

by the 2-out-of-3 property, η_Z is in $we(\mathcal{C})$. So by Lemma 3.2, p is in $fib(\mathcal{C}^{\mathcal{Q}})$. Now, since $\mathcal{C}^{\mathcal{Q}}$ is proper, $p^*\eta_Y$ is in $we(\mathcal{C}^{\mathcal{Q}})$. Again by the 2-out-of-3 property of $\mathcal{C}^{\mathcal{Q}}$, $\tilde{i}$ is in $we(\mathcal{C}^{\mathcal{Q}})$. In particular, by Proposition 2.2, the bottom right square is a homotopy fiber square in $\mathcal{C}$, and since p is in $fib(\mathcal{C})$, the top right square is also a homotopy fiber square in $\mathcal{C}$. Hence, by Proposition 2.3, the total right rectangle is a homotopy fiber square in $\mathcal{C}$. By the naturality of η, the total right rectangle is same as the commutative rectangle,

$$\begin{tikzcd} Z \times_{\mathcal{Q}(Y)} Y \arrow[r, "\tilde{p}"] \arrow[d, "\eta_{Z \times_{\mathcal{Q}(Y)} Y}"'] & Y \arrow[d, "\eta_Y"] \\ \mathcal{Q}(Z \times_{\mathcal{Q}(Y)} Y) \arrow[r, "\mathcal{Q}(\tilde{p})"] \arrow[d, "\mathcal{Q}(p^*\eta_Y)"'] & \mathcal{Q}(Y) \arrow[d, "\eta_{\mathcal{Q}(Y)}"] \\ \mathcal{Q}(Z) \arrow[r, "\mathcal{Q}(p)"'] & \mathcal{Q}(\mathcal{Q}(Y)) \end{tikzcd}$$

Since $p^*\eta_Y$ is in $we(\mathcal{C}^{\mathcal{Q}})$, $\mathcal{Q}(p^*\eta_Y)$ is in $we(\mathcal{C})$. Again by Proposition 2.2, the bottom square is a homotopy fiber square in $\mathcal{C}$. Since the total rectangle is a homotopy fiber square, by Proposition 2.3, the top square is a homotopy fiber square in $\mathcal{C}$ as well. By the 2-out-of-3 property of $\mathcal{C}^{\mathcal{Q}}$, we factorize $\tilde{i} = rk$, k in $we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}})$ and r in $we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}})$, which implies f is a retract of $\tilde{p}r$.

Now, the η-naturality square on r,

$$\begin{tikzcd}
E(\tilde{i}) \arrow[r, "r"] \arrow[d, "\eta_{E(\tilde{i})}"'] & Y \times_{\mathcal{Q}(Y)} Z \arrow[d, "\eta_{Y \times_{\mathcal{Q}(Y)} Z}"] \\
\mathcal{Q}(E(\tilde{i})) \arrow[r, "\mathcal{Q}(r)"'] & \mathcal{Q}(Y \times_{\mathcal{Q}(Y)} Z)
\end{tikzcd}$$

is a homotopy fiber square in $\mathcal{C}$ since r is in $we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}}) = we(\mathcal{C}) \cap fib(\mathcal{C})$ and $\mathcal{Q}(r)$ is in $we(\mathcal{C})$. It follows that the η-naturality square of $\tilde{p}r$,

$$\begin{tikzcd}
E(\tilde{i}) \arrow[r, "r"] \arrow[d, "\eta_{E(\tilde{i})}"'] & Y \times_{\mathcal{Q}(Y)} Z \arrow[r, "\tilde{p}"] \arrow[d, "\eta_{Y \times_{\mathcal{Q}(Y)} Z}"] & Y \arrow[d, "\eta_Y"] \\
\mathcal{Q}(E(\tilde{i})) \arrow[r, "\mathcal{Q}(r)"'] & \mathcal{Q}(Y \times_{\mathcal{Q}(Y)} Z) \arrow[r, "\mathcal{Q}(\tilde{p})"'] & \mathcal{Q}(Y)
\end{tikzcd}$$

is a homotopy fiber square in $\mathcal{C}$. Now, f being a retract of $\tilde{p}r$ in $\mathcal{C}$,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] \arrow[rr, bend left] & Y \arrow[d, "\eta_Y"] \arrow[rrr, bend left] & E(\tilde{i}) \arrow[r, "r"] \arrow[d, "\eta_{E(\tilde{i})}"'] \arrow[rrr, bend left] & Y \times_{\mathcal{Q}(Y)} Z \arrow[r, "\tilde{p}"] \arrow[d, "\eta_{Y \times_{\mathcal{Q}(Y)} Z}"] & Y \arrow[d, "\eta_Y"] \arrow[rr, bend left] & X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
\mathcal{Q}(X) \arrow[r, "\mathcal{Q}(f)"'] \arrow[rr, bend right] & \mathcal{Q}(Y) \arrow[rrr, bend right] & \mathcal{Q}(E(\tilde{i})) \arrow[r, "\mathcal{Q}(r)"'] \arrow[rrr, bend right] & \mathcal{Q}(Y \times_{\mathcal{Q}(Y)} Z) \arrow[r, "\mathcal{Q}(\tilde{p})"'] & \mathcal{Q}(Y) \arrow[rr, bend right] & \mathcal{Q}(X) \arrow[r, "\mathcal{Q}(f)"'] & \mathcal{Q}(Y)
\end{tikzcd}$$

it implies that the η-naturality square on f is a retract of the η-naturality square on $\tilde{p}r$ in $\mathcal{C}^{\square}$. Therefore, by Proposition 2.4, the η-naturality square on f is a homotopy fiber square in $\mathcal{C}$.

Conversely, let f be in $fib(\mathcal{C})$ and that the η-naturality square on f is a homotopy fiber square in $\mathcal{C}$. First we factor $\mathcal{Q}(f) = pi$, where i is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and p is in $fib(\mathcal{C})$. By the proof of Theorem 3.3,

$$\begin{tikzcd}
X \arrow[r, "\tilde{i}"] \arrow[d, "\eta_X"'] & Z \times_{\mathcal{Q}(Y)} Y \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{\eta}"] & Y \arrow[d, "\eta_Y"] \\
\mathcal{Q}(X) \arrow[r, "i"] & Z \arrow[r, "p"] & \mathcal{Q}(Y)
\end{tikzcd}$$

p is in $fib(\mathcal{C}^{\mathcal{Q}})$, which implies $\tilde{p}$ is in $fib(\mathcal{C}^{\mathcal{Q}})$ since $fib(\mathcal{C}^{\mathcal{Q}})$ is closed under taking pullbacks. Now since the η-naturality square on f is a homotopy fiber square in $\mathcal{C}$, $\tilde{i}$ is in $we(\mathcal{C})$. Hence, by the 2-out-of-3 property, $\tilde{i}$ admits a factorization $\tilde{i} = qj$, where j is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and q is in $we(\mathcal{C}) \cap fib(\mathcal{C}) = we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}})$. So we get, $f = \tilde{p}\tilde{i} = (\tilde{p}q)j$, where $(\tilde{p}q)$ is in $fib(\mathcal{C}^{\mathcal{Q}})$. Therefore, by the retract argument, f is a retract of $(\tilde{p}q)$, i.e., in $fib(\mathcal{C}^{\mathcal{Q}})$. $\square$

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