Bousfield-Friedlander theorem

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1 Proper Model Category

In a model category, fibrations are preserved under pullbacks, and cofibrations are preserved under pushouts. But weak equivalences, in general does not have such closure property. In a *proper model category*, weak equivalences are preserved under certain pullbacks and pushouts.

Definition 1.1. (Left proper model category) A model category is called *left proper* if weak equivalences are preserved under pushouts along cofibrations, i.e, for every $f: B \to X$ in $we(\mathcal{C})$ and for every $i: B \to A$ in $cof(\mathcal{C})$, the pushout morphism $i_*f: A \to A \coprod_B X$ is in $we(\mathcal{C})$.

$$\begin{array}{ccc}
A \coprod_{B} X & \longleftarrow & X \\
\downarrow_{i,f} & & \uparrow_{f} \\
A & \longleftarrow & B
\end{array}$$

Definition 1.2. (Right proper model category) A model category is called *right proper* if weak equivalences are preserved under pullbacks along fibrations, i.e, for every $f: A \to Y$ in $we(\mathcal{C})$ and for every $p: X \to Y$ in $fib(\mathcal{C})$, the pullback morphism $p^*f: X \times_Y A \to X$ is in $we(\mathcal{C})$.

$$\begin{array}{ccc}
X \times_Y A & \longrightarrow & A \\
\downarrow^{p^* f} & & \downarrow^f \\
X & \xrightarrow{p} & Y
\end{array}$$

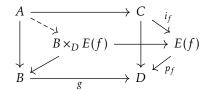
Definition 1.3. (Proper model category) A model category is called *proper* if it is both *left proper* and *right proper*.

2 Homotopy fiber squares

Definition 2.1. Let C be a proper model category. A commutative square in C,

$$\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow f \\
B & \longrightarrow & D
\end{array}$$

is called a *homotopy fiber square* if for some factorization $C \xrightarrow{i_f} E(f) \xrightarrow{p_f} D$ of f where i_f is in we(C) and p_f is in fib(C),



the induced morphism $A \to B \times_D E(f)$ is in $we(\mathcal{C})$.

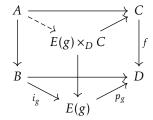
Proposition 2.1. Let \mathcal{C} be a proper model category. Then the following statements are equivalent:-

1. The commutative square in C,

$$\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow f \\
B & \xrightarrow{g} & D
\end{array}$$

is a homotopy fiber square.

2. For some factorization $B \xrightarrow{i_g} E(g) \xrightarrow{p_g} D$ of g where i_g is in $we(\mathcal{C})$ and p_g is in $fib(\mathcal{C})$,



the induced morphism $A \to E(g) \times_D C$ is in we(C).

Proof. (1 \Longrightarrow 2) Suppose the commutative square in (1) is a homotopy fiber square and take any factorization $B \xrightarrow{i_g} E(g) \xrightarrow{p_g} D$ of g where i_g is in $we(\mathcal{C})$ and

 p_g is in fib(C). Then consider the commutative diagram,

$$A \xrightarrow{\hat{i_g}} E(g) \times_D C \xrightarrow{\hat{p_g}} C$$

$$\downarrow \hat{i_f} \qquad \downarrow \hat{i_f} \qquad \downarrow \hat{i_f}$$

$$B \times_D E(f) \xrightarrow{\hat{i_g}} E(g) \times_D E(f) \xrightarrow{\hat{p_g}} E(f)$$

$$\downarrow p_f \qquad \downarrow p_f$$

$$B \xrightarrow{\hat{i_g}} E(g) \xrightarrow{p_g} D$$

where $C \xrightarrow{i_f} E(f) \xrightarrow{p_f} D$ is one such factorization of f, i_f in $we(\mathcal{C})$ and p_f in $fib(\mathcal{C})$, such that $\hat{i_f}$ is in $we(\mathcal{C})$. Now the bottom right square being a pullback, $\tilde{p_g}$ and $\tilde{p_f}$ is in $fib(\mathcal{C})$. Again \mathcal{C} being proper, and the bottom left and top right squares being pullbacks, $\tilde{i_f}$, $\tilde{i_g}$ is in $we(\mathcal{C})$. Finally by the 2-out-of-3 property, $\hat{i_g}$ is in $we(\mathcal{C})$.

 $(2 \implies 1)$ Similar argument as above, just in this case we assume that $\hat{i_g}$ is in $we(\mathcal{C})$, and conclude that $\hat{i_f}$ is in $we(\mathcal{C})$.

Therefore, we conclude that (1) and (2) holds for every such factorization of f and g.

Proposition 2.2. Let C be a proper model category, and consider the commutative square.

$$\begin{array}{ccc}
A & \stackrel{\tilde{f}}{\longrightarrow} & C \\
f \downarrow & & \downarrow g \\
B & \stackrel{f}{\longrightarrow} & D
\end{array}$$

If f is in we(C), then \tilde{f} is in we(C) iff the above square is a homotopy fiber square.

Proof. Let $g = p_g i_g$ be a factorization of p, where i_g is in $we(\mathcal{C})$ and p_g is in $fib(\mathcal{C})$. Consider the commutative diagram,

$$\begin{array}{ccc}
A & \xrightarrow{\tilde{f}} & C \\
\downarrow^{\tilde{i}} & & \downarrow^{i_g} \\
B \times_D E(g) & \xrightarrow{p_g^* f} & E(g) \\
f^* p_g \downarrow & & \downarrow p_g \\
B & \xrightarrow{f} & D
\end{array}$$

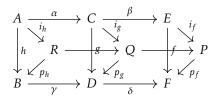
where p_g^*f is in $we(\mathcal{C})$, since \mathcal{C} is proper. Now, if \tilde{f} is in $we(\mathcal{C})$, then by the 2-out-of-3 property, \tilde{i} is in $we(\mathcal{C})$. Conversely, if \tilde{i} is in $we(\mathcal{C})$, then again by the 2-out-of-3 property, \tilde{f} is in $we(\mathcal{C})$.

Proposition 2.3. Let C be a proper model category and consider the commutative square,

$$\begin{array}{ccc}
A & \longrightarrow & C & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & D & \longrightarrow & F
\end{array}$$

where the right square is a homotopy fiber square. Then the left square is also a homotopy fiber square iff the total rectangle is a homotopy fiber square.

Proof. Consider the commutative diagram,



where we first factorize $f=p_fi_f$, where i_f is in $we(\mathcal{C})$, and p_f is in $fib(\mathcal{C})$. Then take the pullback of p_f along δ in order to factorize $g=p_gi_g$. Since p_f is in $fib(\mathcal{C})$, p_g is in $fib(\mathcal{C})$. Also, since the right square is a homotopy fiber square, i_g is in $we(\mathcal{C})$. Again take the pullback of p_g along γ , to factorize $h=p_hi_h$. Now suppose the left square is a homotopy fiber square. Then i_h is in $we(\mathcal{C})$. But p_h is a pullback of p_f along $\delta\gamma$, which implies that the total rectangle is a homotopy fiber square.

Conversely, let the total rectangle be a homotopy fiber square. Then i_h is in $we(\mathcal{C})$, which implies that that the left square in a homotopy fiber square.

Proposition 2.4. Every retract of a homotopy fiber square in \mathcal{C}^{\square} is a homotopy fiber square.

3 \mathcal{Q} -structures on proper model categories

Let \mathcal{C} be a proper model category and $\mathcal{Q}: \mathcal{C} \to \mathcal{C}$ be an endofunctor. A morphism $f: X \to Y$ in \mathcal{C} is called

- a *Q*-equivalence if $Q(f): Q(X) \to Q(Y)$ is a weak equivalence in C.
- a *Q-cofibration* if f is a cofibration in C.
- a *Q-fibration* if *f* has the *right lifting property* with respect to *Q*-trivial cofibrations.

Definition 3.1. (Quillen idempotent monad) Let C be a proper model category. A *Quillen idempotent monad* on C is

• an endofunctor $Q: \mathcal{C} \to \mathcal{C}$

• a natural transformation $\eta: \mathbf{1}_{\mathcal{C}} \to \mathcal{Q}$

such that

- 1. Q is homotopical, i.e., Q preserves weak equivalences.
- 2. For every object X in C, the morphisms, $Q(\eta_X)$, $\eta_{Q(X)}: Q(X) \to Q(Q(X))$ are weak equivalences.
- 3. For a pullback square in C,

$$\begin{array}{ccc}
X \times_Y A & \longrightarrow & A \\
\downarrow^{p^*f} & & & \downarrow^f \\
X & \xrightarrow{p} & Y
\end{array}$$

if p is a Q-fibration and f is a Q-equivalence, then p^*f is a Q-equivalence.

4. For a pushout square in C,

$$\begin{array}{ccc}
A \coprod_{B} X & \longleftarrow & X \\
\downarrow^{i_{*}f} & & \uparrow^{f} \\
A & \longleftarrow & B
\end{array}$$

if i is a Q-cofibration and f is a Q-equivalence, then i_*f is a Q-equivalence.

For a Quillen idempotent monad \mathcal{Q} on \mathcal{C} , let $\mathcal{C}^{\mathcal{Q}}$ denote the category \mathcal{C} equipped with \mathcal{Q} -equivalences, \mathcal{Q} -fibrations and \mathcal{Q} -cofibrations.

Lemma 3.1. A morphism $f: X \to Y$ in C is in $we(C) \cap fib(C)$ iff f is a Q-trivial fibration.

Proof. Let $f: X \to Y$ be in $we(\mathcal{C}) \cap fib(\mathcal{C})$. Then by (1), f is a \mathcal{Q} -equivalence. Now consider the commutative square,

$$\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow f \\
B & \longrightarrow & Y
\end{array}$$

where i is a Q-trivial cofibration. Since i is a Q-cofibration, i is in $cof(\mathcal{C})$ and f being in $we(\mathcal{C}) \cap fib(\mathcal{C})$, f has the right lifting property with respect to i. Hence, f is a Q-trivial fibration.

Conversely, let $f: X \to Y$ in $\mathcal C$ be a $\mathcal Q$ -trivial fibration. Then $f = p_f i_f$ where i_f is in $cof(\mathcal C)$ and p_f is in $we(\mathcal C) \cap fib(\mathcal C)$. Now $\mathcal Q(f) = \mathcal Q(p_f)\mathcal Q(i_f)$, where $\mathcal Q(f)$ is in $we(\mathcal C)$. Also by (1), $\mathcal Q(p_f)$ is in $we(\mathcal C)$. By the 2-out-of-3 property, i_f is a $\mathcal Q$ -equivalence. So, i_f is a $\mathcal Q$ -trivial cofibration, hence has the left lifting property with respect to f. By the retract argument, f is a retract of p_f , therefore, f is in $we(\mathcal C) \cap fib(\mathcal C)$.

Lemma 3.2. If a morphism $f: X \to Y$ is in fib(C), and $\eta_X: X \to Q(X)$, $\eta_Y: Y \to Q(Y)$ are in we(C), then f is a Q-fibration.

Proof. For any commutative square,

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & X \\
\downarrow i & & \downarrow f \\
B & \xrightarrow{\beta} & Y
\end{array}$$

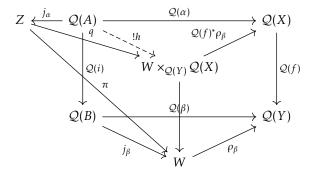
where i is a Q-trivial cofibration, it suffices to show that there exists a lift. First, we factorize the functorial image of the above commutative square,

$$Q(A) \xrightarrow{j_{\alpha}} Z \xrightarrow{\rho_{\alpha}} Q(X)$$

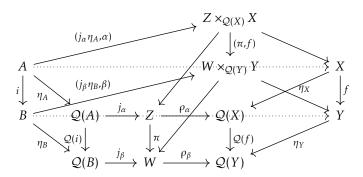
$$Q(i) \downarrow \qquad \qquad \downarrow \pi \qquad \qquad \downarrow Q(f)$$

$$Q(B) \xrightarrow{j_{\beta}} W \xrightarrow{\rho_{\beta}} Q(Y)$$

where, j_{α}, j_{β} are in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and $\rho_{\alpha}, \rho_{\beta}$ are in $fib(\mathcal{C})$. We obtain this factorization in the following way: first factorize $\mathcal{Q}(\beta) = \rho_{\beta}j_{\beta}$, where j_{β} is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and ρ_{β} is in $fib(\mathcal{C})$. Then take the pullback $(\mathcal{Q}(f))^*(\rho_{\beta}) : W \times_{\mathcal{Q}(Y)} \mathcal{Q}(X) \to \mathcal{Q}(X)$ of ρ_{β} along $\mathcal{Q}(f)$, which is in $fib(\mathcal{C})$, since ρ_{β} is. By the universal property, there exists an unique morphism $h: \mathcal{Q}(A) \to W \times_{\mathcal{Q}(Y)} \mathcal{Q}(X)$, such that $\mathcal{Q}(\alpha) = (\mathcal{Q}(f))^*(\rho_{\beta})h$. Again factorize $h = qj_{\alpha}$, where, j_{α} is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and q is in $fib(\mathcal{C})$. Finally, set $\rho_{\alpha} = (\mathcal{Q}(f))^*(\rho_{\beta})q$ and $\pi = (\rho_{\beta})^*(\mathcal{Q}(f))q$.



Now consider the pullback of the right square along the η -naturality square on f ,



we obtain the commutative diagram,

$$A \xrightarrow{(j_{\alpha}\eta_{A},\alpha)} Z \times_{\mathcal{Q}(X)} X \longrightarrow X$$

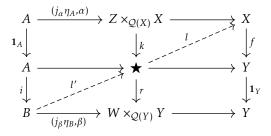
$$\downarrow \downarrow \downarrow (\pi,f) \qquad \qquad \downarrow f$$

$$B \xrightarrow{(j_{\beta}\eta_{B},\beta)} W \times_{\mathcal{Q}(Y)} Y \longrightarrow Y$$

where the left square is due to the universal property of the pullback. Now we show that (π, f) is in we(C). For that, consider the diagram,

$$\begin{array}{cccc} \mathcal{Q}(A) & \stackrel{j_{\alpha}}{\longrightarrow} Z & \stackrel{(\rho_{\alpha})^{*}\eta_{X}}{\longleftarrow} Z \times_{\mathcal{Q}(X)} X \\ \downarrow^{Q(i)} & & \downarrow^{\pi} & \downarrow^{(\pi,f)} \\ \mathcal{Q}(B) & \stackrel{j_{\beta}}{\longrightarrow} W & \stackrel{(\rho_{\beta})^{*}\eta_{Y}}{\longleftarrow} W \times_{\mathcal{Q}(Y)} Y \end{array}$$

Since \mathcal{C} is proper, $(\rho_{\alpha})^*\eta_X$ and $(\rho_{\beta})^*\eta_Y$ are in $we(\mathcal{C})$. By the 2-out-of-3 property, π is in $we(\mathcal{C})$, and which further implies that (π, f) is in $we(\mathcal{C})$. Finally factorize $(\pi, f) = rk$, where k is in $cof(\mathcal{C})$ and r is in $we(\mathcal{C}) \cap fib(\mathcal{C})$. By the 2-out-of-3 property, k is in $we(\mathcal{C}) \cap cof(\mathcal{C})$. Then in the following commutative diagram,



i being in cof(C) and *r* in $we(C) \cap fib(C)$, l' exists. Similarly, *k* being in $we(C) \cap cof(C)$ and *f* in fib(C), l exists. Hence, ll' is our desired lift.

Theorem 3.3. (Bousfield-Friedlander theorem) $C^{\mathbb{Q}}$ is a proper model catgeory, where $we(C^{\mathbb{Q}})$, $cof(C^{\mathbb{Q}})$ and $fib(C^{\mathbb{Q}})$ are \mathbb{Q} -equivalences, \mathbb{Q} -fibrations and \mathbb{Q} -cofibrations respectively.

Proof. Since \mathcal{C} is a model category, $\mathcal{C}^{\mathcal{Q}}$ has limits and colimits. Suppose h = gf, and two of the three morphisms f, g and h are in $we(\mathcal{C}^{\mathcal{Q}})$. Then $\mathcal{Q}(h) = \mathcal{Q}(g)\mathcal{Q}(f)$, and by the 2-out-of-3 property of \mathcal{C} , the third morphism is also in $we(\mathcal{C}^{\mathcal{Q}})$. This proves the 2-out-of-3 property of $\mathcal{C}^{\mathcal{Q}}$. Now, since $cof(\mathcal{C}^{\mathcal{Q}}) = cof(\mathcal{C})$, and by Lemma 2.1, $we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}}) = we(\mathcal{C}) \cap fib(\mathcal{C})$, we have that $(cof(\mathcal{C}^{\mathcal{Q}}), we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}}))$ is a weak factorization system. On the other hand, by the definition of $fib(\mathcal{C}^{\mathcal{Q}})$, we have $fib(\mathcal{C}^{\mathcal{Q}}) = (we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}}))^{|\mathcal{D}|}$.

Now we consider a morphism $f: X \to Y$ in $\mathcal{C}^{\mathbb{Q}}$. Then we factorize $\mathcal{Q}(f)$,

$$Q(X) \xrightarrow{i} Z \xrightarrow{p} Q(Y)$$

i in $we(\mathcal{C}) \cap cof(\mathcal{C})$, hence in $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$ and p in $fib(\mathcal{C})$. In the η -naturality square,

$$\begin{array}{ccc} \mathcal{Q}(X) & \xrightarrow{i} & Z & \xrightarrow{p} & \mathcal{Q}(Y) \\ \eta_{\mathcal{Q}(X)} \downarrow & & & \downarrow \eta_{Z} & & \downarrow \eta_{\mathcal{Q}(Y)} \\ \mathcal{Q}(\mathcal{Q}(X)) & \xrightarrow{\mathcal{Q}(i)} & \mathcal{Q}(Z) & \xrightarrow{\mathcal{Q}(p)} & \mathcal{Q}(\mathcal{Q}(Y)) \end{array}$$

since $\eta_{\mathcal{Q}(X)}$, i and $\mathcal{Q}(i)$ are in $we(\mathcal{C})$, by 2-out-of-3 property, η_Z is in $we(\mathcal{C})$. So by Lemma 2.2, p is in $fib(\mathcal{C}^{\mathcal{Q}})$. Now, we factorize the η -naturality square on f, as the pullback corner morphism \tilde{i} followed by the pullback \tilde{p} of p along η_Y ,

$$X \xrightarrow{\tilde{i}} Z \times_{\mathcal{Q}(Y)} Y \xrightarrow{\tilde{p}} Y$$

$$\downarrow^{\eta_X} \qquad \qquad \downarrow^{\eta_Y} \qquad \qquad \downarrow^{\eta_Y}$$

$$\mathcal{Q}(X) \xrightarrow{i} Z \xrightarrow{p} \mathcal{Q}(Y)$$

By (3), $\tilde{\eta}$ is in $we(\mathcal{C}^{\mathbb{Q}})$, since η_Y is in $we(\mathcal{C}^{\mathbb{Q}})$ and p is in $fib(\mathcal{C}^{\mathbb{Q}})$. By the 2-out-of-3 property, \tilde{i} is in $we(\mathcal{C}^{\mathbb{Q}})$. Also, since p is in $fib(\mathcal{C}^{\mathbb{Q}})$ and $fib(\mathcal{C}^{\mathbb{Q}}) = (we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}}))^{\mathbb{Z}}$, \tilde{p} is in $fib(\mathcal{C}^{\mathbb{Q}})$. Finally, factorize $\tilde{i} = \tilde{q}\tilde{j}$, where \tilde{j} is in $cof(\mathcal{C}^{\mathbb{Q}})$ and \tilde{q} is in $we(\mathcal{C}^{\mathbb{Q}}) \cap fib(\mathcal{C}^{\mathbb{Q}})$. By the 2-out-of-3 property of $\mathcal{C}^{\mathbb{Q}}$, \tilde{j} is in $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$ and $\tilde{p}\tilde{q}$ is our required morphism in $fib(\mathcal{C}^{\mathbb{Q}})$. Let j be a morphism in $\mathbb{Z}^{\mathbb{Q}}$, and factorize j = rk, k in $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$ and r in $fib(\mathcal{C}^{\mathbb{Q}})$. By the retract argument, j is a retract of k. But, $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$ is closed under retracts, since $cof(\mathcal{C}^{\mathbb{Q}}) = cof(\mathcal{C})$ and $we(\mathcal{C}) \subseteq we(\mathcal{C}^{\mathbb{Q}})$ are closed under retracts. Therefore, $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}}) = \mathbb{Z}^{\mathbb{Q}}$ fib $(\mathcal{C}^{\mathbb{Q}})$ and $(we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$, $fib(\mathcal{C}^{\mathbb{Q}})$) is a weak factorization system. The properness of $\mathcal{C}^{\mathbb{Q}}$ follows from (3) and (4).

Proposition 3.4. A morphism $f: X \to Y$ in C is a Q-fibration iff f is in fib(C)

and the η -naturality square on f,

$$X \xrightarrow{\eta_X} \mathcal{Q}(X)$$

$$f \downarrow \qquad \qquad \downarrow \mathcal{Q}(f)$$

$$Y \xrightarrow{\eta_Y} \mathcal{Q}(Y)$$

is a homotopy fiber square in C.

Proof. Let $f: X \to Y$ in \mathcal{C} be a \mathcal{Q} -fibration. Then by the definition of a \mathcal{Q} -fibration, f is in $fib(\mathcal{C})$. We factorize $\mathcal{Q}(f) = pi$, where i is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and p is in $fib(\mathcal{C})$, and then consider the commutative diagram,

by the 2-out-of-3 property, η_Z is in $we(\mathcal{C})$. So by Lemma 3.2, p is in $fib(\mathcal{C}^Q)$. Now, since \mathcal{C}^Q is proper, $p^*\eta_Y$ is in $we(\mathcal{C}^Q)$. Again by the 2-out-of-3 property of \mathcal{C}^Q , \tilde{i} is in $we(\mathcal{C}^Q)$. In particular, by Proposition 2.2, the bottom right square is a homotopy fiber square in \mathcal{C} , and since p is in $fib(\mathcal{C})$, the top right square is also a homotopy fiber square in \mathcal{C} . Hence, by Proposition 2.3, the total right rectangle is a homotopy fiber square in \mathcal{C} . By the naturality of η , the total right rectangle is same as the commutative rectangle,

$$Z \times_{\mathcal{Q}(Y)} Y \xrightarrow{\tilde{p}} Y$$

$$\eta_{Z \times_{\mathcal{Q}(Y)} Y} \downarrow \qquad \qquad \downarrow \eta_{Y}$$

$$\mathcal{Q}(Z \times_{\mathcal{Q}(Y)} Y) \xrightarrow{\mathcal{Q}(\tilde{p})} \mathcal{Q}(Y)$$

$$\mathcal{Q}(p^{*}\eta_{Y}) \downarrow \qquad \qquad \downarrow \eta_{\mathcal{Q}(Y)}$$

$$\mathcal{Q}(Z) \xrightarrow{\mathcal{Q}(p)} \mathcal{Q}(\mathcal{Q}(Y))$$

Since $p^*\eta_Y$ is in $we(\mathcal{C}^\mathbb{Q})$, $\mathcal{Q}(p^*\eta_Y)$ is in $we(\mathcal{C})$. Again by Proposition 2.2, the bottom square is a homotopy fiber square in \mathcal{C} . Since the total rectangle is a homotopy fiber square, by Proposition 2.3, the top square is a homotopy fiber square in \mathcal{C} as well. By the 2-out-of-3 property of $\mathcal{C}^\mathbb{Q}$, we factorize $\tilde{i} = rk$, k in $we(\mathcal{C}^\mathbb{Q}) \cap cof(\mathcal{C}^\mathbb{Q})$ and r in $we(\mathcal{C}^\mathbb{Q}) \cap fib(\mathcal{C}^\mathbb{Q})$, which implies f is a retract of $\tilde{p}r$.

Now, the η -naturality square on r,

$$E(\tilde{i}) \xrightarrow{r} Y \times_{Q(Y)} Z$$

$$\downarrow^{\eta_{E(\tilde{i})}} \qquad \qquad \downarrow^{\eta_{Y} \times_{Q(Y)} Z}$$

$$Q(E(\tilde{i})) \xrightarrow{Q(r)} Q(Y \times_{Q(Y)} Z)$$

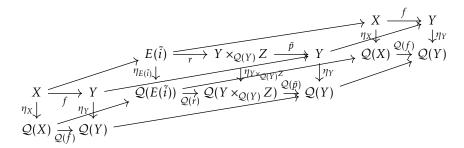
is a homotopy fiber square in $\mathcal C$ since r is in $we(\mathcal C^{\mathcal Q}) \cap fib(\mathcal C^{\mathcal Q}) = we(\mathcal C) \cap fib(\mathcal C)$ and $\mathcal Q(r)$ is in $we(\mathcal C)$. It follows that the η -naturality square of $\tilde pr$,

$$E(\tilde{i}) \xrightarrow{r} Y \times_{\mathcal{Q}(Y)} Z \xrightarrow{\tilde{p}} Y$$

$$\downarrow^{\eta_{E(\tilde{i})}} \qquad \qquad \downarrow^{\eta_{Y} \times_{\mathcal{Q}(Y)} Z} \qquad \downarrow^{\eta_{Y}}$$

$$\mathcal{Q}(E(\tilde{i})) \xrightarrow{\mathcal{Q}(r)} \mathcal{Q}(Y \times_{\mathcal{Q}(Y)} Z) \xrightarrow{\mathcal{Q}(\tilde{p})} \mathcal{Q}(Y)$$

is a homotopy fiber square in C. Now, f being a retract of $\tilde{p}r$ in C,



it implies that the η -naturality square on f is a retract of the η -naturality square on $\tilde{p}r$ in \mathcal{C}^{\square} . Therefore, by Proposition 2.4, the η -naturality square on f is a homotopy fiber square in \mathcal{C} .

Conversely, let f be in $fib(\mathcal{C})$ and that the η -naturality square on f is a homotopy fiber square in \mathcal{C} . First we factor $\mathcal{Q}(f) = pi$, where i is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and p is in $fib(\mathcal{C})$. By the proof of Theorem 3.3,

$$X \xrightarrow{\tilde{i}} Z \times_{Q(Y)} Y \xrightarrow{\tilde{p}} Y$$

$$\uparrow_{\eta_X} \downarrow \qquad \qquad \downarrow_{\eta_{\tilde{i}}} \qquad \qquad \downarrow_{\eta_{Y}}$$

$$Q(X) \xrightarrow{i} Z \xrightarrow{p} Q(Y)$$

p is in $fib(\mathcal{C}^{\mathcal{Q}})$, which implies \tilde{p} is in $fib(\mathcal{C}^{\mathcal{Q}})$ since $fib(\mathcal{C}^{\mathcal{Q}})$ is closed under taking pullbacks. Now since the η -naturality square on f is a homotopy fiber square in \mathcal{C} , \tilde{i} is in $we(\mathcal{C})$. Hence, by the 2-out-of-3 property, \tilde{i} admits a factorization $\tilde{i}=qj$, where j is in $we(\mathcal{C}) \cap cof(\mathcal{C})$ and q is in $we(\mathcal{C}) \cap fib(\mathcal{C}) = we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}})$. So we get, $f = \tilde{p}\tilde{i} = (\tilde{p}q)j$, where $(\tilde{p}q)$ is in $fib(\mathcal{C}^{\mathcal{Q}})$. Therefore, by the retract argument, f is a retract of $(\tilde{p}q)$, i.e., in $fib(\mathcal{C}^{\mathcal{Q}})$.

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