

The Morita $(\infty, 2)$ -category of a monoidal category as a 2-complicial set¹

Arghan Dutta

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University of Ottawa, Canada

¹j/w Stefano Luneia, Martina Rovelli and Sam Silver

Let $\mathcal{C}^\otimes$ be a monoidal category + “assumptions” [Vit92].

$$\left\{ \begin{array}{c} \text{Monoids} \\ A, B, C, \dots \end{array} \right\}_0 + \left\{ \begin{array}{c} \text{Bimodules} \\ A \xrightarrow{M} B \end{array} \right\}_1 + \left\{ \begin{array}{c} \text{Bimodule maps} \\ \begin{array}{ccc} & M & \\ A & \begin{array}{c} \Downarrow \varphi \end{array} & B \\ & N & \end{array} \end{array} \right\}_2$$

form a bicategory $\text{Alg}^{bi}(\mathcal{C}^\otimes)$, the *Morita bicategory* of $\mathcal{C}^\otimes$.

The technical assumptions are that $\mathcal{C}^\otimes$ must contain some coequalizers, and that they are preserved by $(-) \otimes Z$ for all $Z \in \mathcal{C}^\otimes$.

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Examples: $\mathbf{Ab}^\otimes$ (Rings and Bimodules), $\mathbf{Set}^\Pi$ (Sets and cospans). More generally, any cocomplete closed monoidal categories.

Balanced tensor product

For an (A, B) -bimodule M , and (B, C) -bimodule N , the b.t.p is the

$$M \otimes_B N := \operatorname{coeq} \left[(M \otimes B) \otimes N \begin{array}{c} \xrightarrow{r_M \otimes N} \\ \xrightarrow{\alpha_{M,B,N} \bullet (M \otimes \ell'_N)} \end{array} M \otimes N \right]$$

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The universal property of coequalizers induce horizontal composition

$$\begin{array}{c} \begin{array}{ccccc} & M & & N & \\ & \curvearrowright & & \curvearrowright & \\ A & \Downarrow \varphi & B & \Downarrow \psi & C \\ & \curvearrowleft & & \curvearrowleft & \\ & M' & & N' & \end{array} & \rightsquigarrow & \begin{array}{ccc} & M \otimes_B N & \\ & \curvearrowright & \\ A & \Downarrow \varphi \otimes_B \psi & B \\ & \curvearrowleft & \\ & M' \otimes_B N' & \end{array} \end{array}$$

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$$\left\{ \begin{array}{c} \text{Monoids} \\ A, B, C, \dots \end{array} \right\}_0 + \left\{ \begin{array}{c} \text{Bimodules} \\ A \xrightarrow{M} B \end{array} \right\}_1 + \left\{ \begin{array}{c} \text{Bimodule maps} \\ \begin{array}{ccc} & B & \\ M \nearrow & \Downarrow \varphi & \searrow N \\ A & \xrightarrow{P} & C \end{array} \end{array} \right\}_2 + \left\{ \begin{array}{c} \text{coskeletal} \\ \text{extension} \end{array} \right\}_{\geq 3}$$

form an $(\infty, 2)$ -category $\mathfrak{M}^{\flat}(\mathcal{C}^\otimes)$, the *Morita* ∞ -bicategory of $\mathcal{C}^\otimes$.

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form an $(\infty, 2)$ -category $\mathfrak{M}^{\natural}(\mathcal{C}^\otimes)$, the *Morita ∞ -bicategory* of $\mathcal{C}^\otimes$.

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Theorem: $\mathfrak{M}^{\natural}(\mathcal{C}^\otimes)$ is a 2-complicial set.

3-simplex

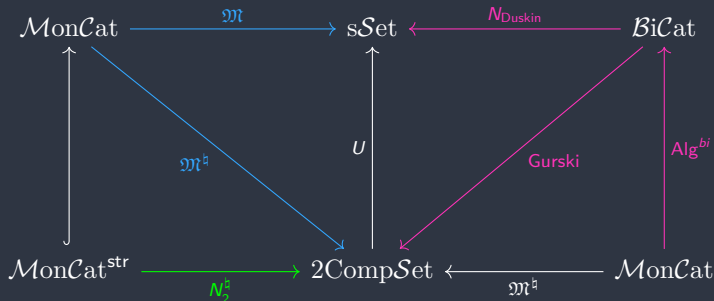
A 3-simplex in $\mathfrak{M}^b(\mathcal{C}^\otimes)$ will be depicted as



such that

$$(\varphi_{012} \otimes_{A_2} M_{23}) \bullet \varphi_{023} = \overline{\alpha}_{M_{01}|M_{12}|M_{23}} \bullet (M_{01} \otimes_{A_1} \varphi_{123}) \bullet \varphi_{013}$$

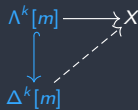
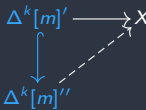
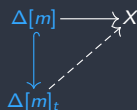
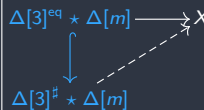
Comparable existing work



- ▶ Nick's approach [Gur09].
- ▶ DLRS approach [DLRS25].
- ▶ Ozornova-Rovelli natural nerve [OR21].

Complcial sets

An n -complcial set [Ver08, Rie18, OR20] is a marked simplicial set X that has RLP w.r.t

Horn	Thinness	Triviality	Saturation
 $ \begin{array}{ccc} \Lambda^k[m] & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^k[m] & & \end{array} $	 $ \begin{array}{ccc} \Delta^k[m]' & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^k[m]'' & & \end{array} $	 $ \begin{array}{ccc} \Delta[m] & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \text{dashed} & \\ \Delta[m]_t & & \end{array} $	 $ \begin{array}{ccc} \Delta[3]^{\text{eq}} \star \Delta[m] & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \text{dashed} & \\ \Delta[3]^{\#} \star \Delta[m] & & \end{array} $
$m \geq 1, \quad 0 \leq k \leq m$	$m \geq 2, \quad 0 \leq k \leq m$	$m > n$	$m \geq -1$
Coherent composition of cells, and marked has inverse.	Marked has 2-out-of-3 property.	Higher cells are equivalences.	Cells that have an inverse are marked.

Marked monoidal nerve $\mathfrak{M}^{\natural}$

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2. The set $\mathfrak{M}_2^{\text{mk}}(\mathcal{C}^\otimes)$ of marked 2-simplices is given by the 2-simplices inhabited by an isomorphism, i.e.

$$\mathfrak{M}_2^{\text{mk}}(\mathcal{C}^\otimes) := \left\{ \begin{array}{ccc} & A_1 & \\ M_{01} \nearrow & & \searrow M_{12} \\ A_0 & \xrightarrow{M_{02}} & A_2 \end{array} \quad \Downarrow \cong \right\}$$

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3. The set $\mathfrak{M}_1^{\text{mk}}(\mathcal{C}^\otimes)$ of marked 1-simplices is given by the bimodules which are equivalences in the following sense.

$$\mathfrak{M}_1^{\text{mk}}(\mathcal{C}^\otimes) := \left\{ {}_A M_B \left| \begin{array}{ccc} & B & \\ M \nearrow & \Downarrow \cong & \searrow M'' \\ A & \xrightarrow{A} & A \end{array} \quad \begin{array}{ccc} & A & \\ M' \nearrow & \Downarrow \cong & \searrow M \\ B & \xrightarrow{B} & B \end{array} \right. \right\}$$

Our approach

- ▶ Rather than directly verifying the bicategory axioms, we leverage the combinatorics of simplicial sets to reformulate these coherence conditions.
- ▶ Offers a proof of concept for explicitly and efficiently encoding higher categorical structure.
- ▶ Offers a pathway toward scaling these methods to even higher dimensions, such as when treating braided (work in progress) or symmetric monoidal categories.

Thank You!

For your time and attention.

Horns

Proposition: *The marked simplicial set $\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$ has the RLP with respect to the map $\Lambda^k[m] \rightarrow \Delta^k[m]$ for $m > 4$ and $0 \leq k \leq m$.*

Since $\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$ is 3-coskeletal, we have

$$\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes}) \cong \text{cosk}_3 \text{tr}_3 \mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$$

$$\begin{array}{ccc} \Lambda^k[m] & \longrightarrow & \mathfrak{M}^{\natural}(\mathcal{C}^{\otimes}) \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^k[m] & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \text{tr}_3 \Lambda^k[m] & \longrightarrow & \text{tr}_3 \mathfrak{M}^{\natural}(\mathcal{C}^{\otimes}) \\ \downarrow \cong & \nearrow \text{dashed} & \\ \text{tr}_3 \Delta^k[m] & & \end{array}$$

Therefore, it suffices to solve by hand the complicial horn lifting problems corresponding to $m = 1, 2, 3, 4$.

Thinness

Proposition: The marked simplicial set $\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$ has the RLP with respect to the map $\Delta^k[m]' \rightarrow \Delta^k[m]''$ for $m > 3$ and $0 \leq k \leq m$.

$\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$ is 2-trivial, and moreover we have the adjunction

$$\mathrm{th}_3: \mathrm{msSet} \rightleftarrows \mathrm{msSet}: \mathrm{sp}_3$$

$$\begin{array}{ccc} \Delta^k[m]' \longrightarrow \mathfrak{M}^{\natural}(\mathcal{C}^{\otimes}) & & \mathrm{th}_3 \Delta^k[m]' \longrightarrow \mathfrak{M}^{\natural}(\mathcal{C}^{\otimes}) \\ \downarrow & \nearrow \text{dashed} & \downarrow \cong \\ \Delta^k[m]'' & & \mathrm{th}_3 \Delta^k[m]'' \end{array} \quad \rightleftarrows$$

Therefore, it suffices to solve by hand the complicial thinness lifting problems corresponding to $m = 1, 2, 3$.

Proposition: *The marked simplicial set $\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$ has the RLP with respect to the map $\Delta[3]^{eq} \star \Delta[\ell] \rightarrow \Delta[3]^{\sharp} \star \Delta[\ell]$ for $\ell > 0$.*

$$\begin{array}{ccc}
 \Delta[3]^{eq} \star \Delta[\ell] & \longrightarrow & \mathfrak{M}^{\natural}(\mathcal{C}^{\otimes}) \\
 \downarrow & \nearrow & \\
 \Delta[3]^{eq} \star \Delta[\ell] & &
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccc}
 \mathrm{th}_3(\Delta[3]^{eq} \star \Delta[\ell]) & \longrightarrow & \mathfrak{M}^{\natural}(\mathcal{C}^{\otimes}) \\
 \downarrow \cong & \nearrow & \\
 \mathrm{th}_3(\Delta[3]^{eq} \star \Delta[\ell]) & &
 \end{array}$$

Therefore, it suffices to solve by hand the complicial thinness lifting problems corresponding to $\ell = -1, 0$.

References I

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