The Morita $(\infty, 2)$-category of a monoidal category as a 2-complicial set[1]

Arghan Dutta
CT2026 Baltimore, USA, July, 2026

University of Ottawa, Canada

[1] j/w Stefano Luneia, Martina Rovelli and Sam Silver

Folklore

Let $\mathcal{C}^{\otimes}$ be a monoidal category + "assumptions" [Vit92].

$$\left\{\begin{matrix}\text{Monoids} \\ A, B, C, \cdots\end{matrix}\right\}_0 + \left\{\begin{matrix}\text{Bimodules} \\ A \xrightarrow{M} B\end{matrix}\right\}_1 + \left\{\begin{matrix}\text{Bimodule maps} \\ \begin{tikzcd} A \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \end{tikzcd}\end{matrix}\right\}_2$$

form a bicategory $\mathsf{Alg}^{bi}(\mathcal{C}^{\otimes})$, the *Morita bicategory* of $\mathcal{C}^{\otimes}$.

The technical assumptions are that $\mathcal{C}^{\otimes}$ must contain some coequalizers, and that they are preserved by $(-) \otimes Z$ for all $Z \in \mathcal{C}^{\otimes}$.

Folklore

Let $\mathcal{C}^{\otimes}$ be a monoidal category + "assumptions" [Vit92].

$$\left\{\begin{matrix}\text{Monoids}\\ A, B, C, \cdots\end{matrix}\right\}_0 + \left\{\begin{matrix}\text{Bimodules}\\ A \xrightarrow{M} B\end{matrix}\right\}_1 + \left\{\begin{matrix}\text{Bimodule maps}\\ \begin{tikzcd} A \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \end{tikzcd}\end{matrix}\right\}_2$$

form a bicategory $\mathrm{Alg}^{bi}(\mathcal{C}^{\otimes})$, the *Morita bicategory* of $\mathcal{C}^{\otimes}$.

The technical assumptions are that $\mathcal{C}^{\otimes}$ must contain some coequalizers, and that they are preserved by $(-) \otimes Z$ for all $Z \in \mathcal{C}^{\otimes}$.

Examples: $\mathbf{Ab}^{\otimes}$ (Rings and Bimodules), $\mathbf{Set}^{\amalg}$(Sets and cospans). More generally, any cocomplete closed monoidal categories.

Balanced tensor product

For an (A, B)-bimodule M, and (B, C)-bimodule N, the b.t.p is the

$$M \otimes_B N := \mathrm{coeq}\left[\begin{tikzcd} (M \otimes B) \otimes N \arrow[r, shift left, "r_M \otimes N"] \arrow[r, shift right, "\alpha_{M,B,N} \bullet (M \otimes \ell_N)"'] & M \otimes N \end{tikzcd}\right]$$

Balanced tensor product

For an (A, B)-bimodule M, and (B, C)-bimodule N, the b.t.p is the

$$M \otimes_B N := \mathrm{coeq}\left[\begin{tikzcd} (M \otimes B) \otimes N \arrow[r, shift left, "r_M \otimes N"] \arrow[r, shift right, "\alpha_{M,B,N} \bullet (M \otimes \ell_N)"'] & M \otimes N \end{tikzcd}\right]$$

The universal property of coequalizers induce horizontal composition

$$\begin{tikzcd} A \arrow[r, bend left, "M"{name=U1}] \arrow[r, bend right, "M'"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "\varphi"] & B \arrow[r, bend left, "N"{name=U2}] \arrow[r, bend right, "N'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\psi"] & C \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left, "M \otimes_B N"{name=U}] \arrow[r, bend right, "M' \otimes_B N'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\varphi \otimes_B \psi"] & B \end{tikzcd}$$

DLRS

Let $\mathcal{C}^{\otimes}$ be a monoidal category + "assumptions" [DLRS25].

$$\left\{\begin{matrix}\text{Monoids}\\ A, B, C, \cdots\end{matrix}\right\}_0 + \left\{\begin{matrix}\text{Bimodules}\\ A \xrightarrow{M} B\end{matrix}\right\}_1 + \left\{\begin{matrix}\text{Bimodule maps}\\ \begin{tikzcd} & B \arrow[dr, "N"] \arrow[d, phantom, "\varphi\Downarrow"] & \\ A \arrow[ur, "M"] \arrow[rr, "P"'] & & C \end{tikzcd}\end{matrix}\right\}_2 + \left\{\begin{matrix}\text{coskeletal}\\ \text{extension}\end{matrix}\right\}_{\geq 3}$$

form an $(\infty, 2)$-category $\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$, the *Morita ∞-bicategory* of $\mathcal{C}^{\otimes}$.

The n-coskeleton extends an n-truncated simplicial set by adding only the higher simplices uniquely forced by the lower-dimensional boundary data.

DLRS

Let $\mathcal{C}^{\otimes}$ be a monoidal category + "assumptions" [DLRS25].

$$\left\{\begin{matrix}\text{Monoids}\\ A, B, C, \cdots\end{matrix}\right\}_0 + \left\{\begin{matrix}\text{Bimodules}\\ A \xrightarrow{M} B\end{matrix}\right\}_1 + \left\{\begin{matrix}\text{Bimodule maps}\\ \begin{tikzcd} & B \arrow[dr, "N"] \arrow[d, phantom, "\varphi\Downarrow"] & \\ A \arrow[ur, "M"] \arrow[rr, "P"'] & {} & C \end{tikzcd}\end{matrix}\right\}_2 + \left\{\begin{matrix}\text{coskeletal}\\ \text{extension}\end{matrix}\right\}_{\geq 3}$$

form an $(\infty, 2)$-category $\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$, the *Morita ∞-bicategory* of $\mathcal{C}^{\otimes}$.

The n-coskeleton extends an n-truncated simplicial set by adding only the higher simplices uniquely forced by the lower-dimensional boundary data.

Theorem: $\mathfrak{M}^{\natural}(\mathcal{C}^{\otimes})$ *is a 2-complicial set.*

3-simplex

A 3-simplex in $\mathfrak{M}^\natural(\mathcal{C}^\otimes)$ will be depicted as

$$\begin{tikzcd} A_1 \arrow[r, "M_{12}"] \arrow[dr, phantom, "\varphi_{012}"] & A_2 \arrow[d, "M_{23}"] \\ A_0 \arrow[u, "M_{01}"] \arrow[ur, "A_{02}"] \arrow[r, "M_{01}"'] & A_3 \arrow[ul, phantom, "\varphi_{023}"] \end{tikzcd} \quad \Rrightarrow \quad \begin{tikzcd} A_1 \arrow[r, "M_{12}"] \arrow[dr, "A_{13}"] \arrow[dr, phantom, "\varphi_{013}"'] & A_2 \arrow[d, "M_{23}"] \arrow[dl, phantom, "\varphi_{123}"] \\ A_0 \arrow[u, "M_{01}"] \arrow[r, "M_{03}"'] & A_3 \end{tikzcd}$$

such that

$$(\varphi_{012} \otimes_{A_2} M_{23}) \bullet \varphi_{023} = \overline{\alpha}_{M_{01}|M_{12}|M_{23}} \bullet (M_{01} \otimes_{A_1} \varphi_{123}) \bullet \varphi_{013}$$

Comparable existing work

$$\begin{tikzcd}
\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \arrow[rr, "\mathfrak{M}"] \arrow[drr, "\mathfrak{M}^\natural"] & & \mathrm{s}\mathcal{S}\mathrm{et} & & \mathcal{B}\mathrm{i}\mathcal{C}\mathrm{at} \arrow[ll, "N_{\mathsf{Duskin}}"'] \arrow[dll, "\mathsf{Gurski}"] \\
\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}^{\mathsf{str}} \arrow[u, hook] \arrow[rr, "N_2^\natural"'] & & 2\mathrm{Comp}\mathcal{S}\mathrm{et} \arrow[u, "U"] & & \mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \arrow[ll, "\mathfrak{M}^\natural"] \arrow[u, "\mathsf{Alg}^{bi}"']
\end{tikzcd}$$

- Nick's approach [Gur09].
- DLRS approach [DLRS25].
- Ozornova-Rovelli natural nerve [OR21].

Complicial sets

An *n-complicial set* [Ver08, Rie18, OR20] is a marked simplicial set X that has RLP w.r.t

Horn	**Thinness**	**Triviality**	**Saturation**
$\begin{tikzcd} \Lambda^k[m] \arrow[r] \arrow[d, hook] & X \\ \Delta^k[m] \arrow[ur, dashed] & \end{tikzcd}$	$\begin{tikzcd} \Delta^k[m]' \arrow[r] \arrow[d, hook] & X \\ \Delta^k[m]'' \arrow[ur, dashed] & \end{tikzcd}$	$\begin{tikzcd} \Delta[m] \arrow[r] \arrow[d, hook] & X \\ \Delta[m]_t \arrow[ur, dashed] & \end{tikzcd}$	$\begin{tikzcd} \Delta[3]^{\mathrm{eq}} \star \Delta[m] \arrow[r] \arrow[d, hook] & X \\ \Delta[3]^\sharp \star \Delta[m] \arrow[ur, dashed] & \end{tikzcd}$
$m \geq 1, \quad 0 \leq k \leq m$	$m \geq 2, \quad 0 \leq k \leq m$	$m > n$	$m \geq -1$
Coherent composition of cells, and marked has inverse.	Marked has 2-out-of-3 property.	Higher cells are equivalences.	Cells that have an inverse are marked.

Marked monoidal nerve $\mathfrak{M}^\natural$

Marked monoidal nerve $\mathfrak{M}^\natural$

1. For $k > 2$, every k-simplex is marked, i.e. $\mathfrak{M}_k^{\mathrm{mk}}(\mathcal{C}^\otimes) = \mathfrak{M}_k(\mathcal{C}^\otimes)$.

Marked monoidal nerve $\mathfrak{M}^\natural$

1. For $k > 2$, every k-simplex is marked, i.e. $\mathfrak{M}_k^{\mathrm{mk}}(\mathcal{C}^\otimes) = \mathfrak{M}_k(\mathcal{C}^\otimes)$.
2. The set $\mathfrak{M}_2^{\mathrm{mk}}(\mathcal{C}^\otimes)$ of marked 2-simplices is given by the 2-simplices inhabited by an isomorphism, i.e.

$$\mathfrak{M}_2^{\mathrm{mk}}(\mathcal{C}^\otimes) := \left\{ \begin{tikzcd} & A_1 \arrow[dr, "M_{12}"] \arrow[d, phantom, "\Downarrow\cong"] & \\ A_0 \arrow[ur, "M_{01}"] \arrow[rr, "M_{02}"'] & {} & A_2 \end{tikzcd} \right\}$$

Marked monoidal nerve $\mathfrak{M}^{\natural}$

1. For $k > 2$, every k-simplex is marked, i.e. $\mathfrak{M}_k^{\mathrm{mk}}(\mathcal{C}^{\otimes}) = \mathfrak{M}_k(\mathcal{C}^{\otimes})$.
2. The set $\mathfrak{M}_2^{\mathrm{mk}}(\mathcal{C}^{\otimes})$ of marked 2-simplices is given by the 2-simplices inhabited by an isomorphism, i.e.

$$\mathfrak{M}_2^{\mathrm{mk}}(\mathcal{C}^{\otimes}) := \left\{ \begin{tikzcd} & A_1 \arrow[dr, "M_{12}"] \arrow[d, phantom, "\Downarrow\cong"] & \\ A_0 \arrow[ur, "M_{01}"] \arrow[rr, "M_{02}"'] & {} & A_2 \end{tikzcd} \right\}$$

3. The set $\mathfrak{M}_1^{\mathrm{mk}}(\mathcal{C}^{\otimes})$ of marked 1-simplices is given by the bimodules which are equivalences in the following sense.

$$\mathfrak{M}_1^{\mathrm{mk}}(\mathcal{C}^{\otimes}) := \left\{ {}_A M_B \;\middle|\; \begin{tikzcd} & B \arrow[dr, "M''"] \arrow[d, phantom, "\Downarrow\cong"] & \\ A \arrow[ur, "M"] \arrow[rr, "A"'] & {} & A \end{tikzcd} \quad \begin{tikzcd} & A \arrow[dr, "M"] \arrow[d, phantom, "\Downarrow\cong"] & \\ B \arrow[ur, "M'"] \arrow[rr, "B"'] & {} & B \end{tikzcd} \right\}$$

Our approach

- Rather than directly verifying the bicategory axioms, we leverage the combinatorics of simplicial sets to reformulate these coherence conditions.
- Offers a proof of concept for explicitly and efficiently encoding higher categorical structure.
- Offers a pathway toward scaling these methods to even higher dimensions, such as when treating braided (work in progress) or symmetric monoidal categories.

Thank You!

For your time and attention.

Horns

Proposition: *The marked simplicial set* $\mathfrak{M}^\natural(\mathcal{C}^\otimes)$ *has the RLP with respect to the map* $\Lambda^k[m] \to \Delta^k[m]$ *for* $m > 4$ *and* $0 \leq k \leq m$.

Since $\mathfrak{M}^\natural(\mathcal{C}^\otimes)$ is 3-coskeletal, we have

$$\mathfrak{M}^\natural(\mathcal{C}^\otimes) \cong \mathrm{cosk}_3 \mathrm{tr}_3 \mathfrak{M}^\natural(\mathcal{C}^\otimes)$$

$$\begin{tikzcd} \Lambda^k[m] \arrow[r] \arrow[d] & \mathfrak{M}^\natural(\mathcal{C}^\otimes) \\ \Delta^k[m] \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathrm{tr}_3\Lambda^k[m] \arrow[r] \arrow[d, "\cong"'] & \mathrm{tr}_3\mathfrak{M}^\natural(\mathcal{C}^\otimes) \\ \mathrm{tr}_3\Delta^k[m] \arrow[ur, dashed] & \end{tikzcd}$$

Therefore, it suffices to solve by hand the complicial horn lifting problems corresponding to $m = 1, 2, 3, 4$.

Thinness

Proposition: *The marked simplicial set $\mathfrak{M}^\natural(\mathcal{C}^\otimes)$ has the RLP with respect to the map $\Delta^k[m]' \to \Delta^k[m]''$ for $m > 3$ and $0 \leq k \leq m$.*

$\mathfrak{M}^\natural(\mathcal{C}^\otimes)$ is 2-trivial, and moreover we have the adjunction

$$\mathrm{th}_3 \colon msSet \rightleftarrows msSet \colon \mathrm{sp}_3$$

$$\begin{tikzcd} \Delta^k[m]' \arrow[r] \arrow[d] & \mathfrak{M}^\natural(\mathcal{C}^\otimes) \\ \Delta^k[m]'' \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathrm{th}_3\Delta^k[m]' \arrow[r] \arrow[d, "\cong"'] & \mathfrak{M}^\natural(\mathcal{C}^\otimes) \\ \mathrm{th}_3\Delta^k[m]'' \arrow[ur, dashed] & \end{tikzcd}$$

Therefore, it suffices to solve by hand the complicial thinness lifting problems corresponding to $m = 1, 2, 3$.

Saturation

Proposition: *The marked simplicial set* $\mathfrak{M}^\natural(\mathcal{C}^\otimes)$ *has the RLP with respect to the map* $\Delta[3]^{eq} \star \Delta[\ell] \to \Delta[3]^\sharp \star \Delta[\ell]$ *for* $\ell > 0$.

$$\begin{tikzcd} \Delta[3]^{eq} \star \Delta[\ell] \arrow[r] \arrow[d] & \mathfrak{M}^\natural(\mathcal{C}^\otimes) \\ \Delta[3]^{eq} \star \Delta[\ell] \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathrm{th}_3(\Delta[3]^{eq} \star \Delta[\ell]) \arrow[r] \arrow[d, "\cong"'] & \mathfrak{M}^\natural(\mathcal{C}^\otimes) \\ th_3(\Delta[3]^{eq} \star \Delta[\ell]) \arrow[ur, dashed] & \end{tikzcd}$$

Therefore, it suffices to solve by hand the complicial thinness lifting problems corresponding to $\ell = -1, 0$.

References I

Arghan Dutta, Stefano Luneia, Martina Rovelli, and Sam Silver, *The morita $(\infty, 2)$-category of a monoidal category as a 2-complicial set*, 2025.

Nick Gurski, *Nerves of bicategories as stratified simplicial sets*, J. Pure Appl. Algebra **213** (2009), no. 6, 927–946. MR 2498786

Viktoriya Ozornova and Martina Rovelli, *Model structures for (∞, n)-categories on (pre)stratified simplicial sets and prestratified simplicial spaces*, Algebr. Geom. Topol. **20** (2020), no. 3, 1543–1600. MR 4105558

______, *Nerves of 2-categories and 2-categorification of $(\infty, 2)$-categories*, Adv. Math. **391** (2021), Paper No. 107948, 39. MR 4301487

References II

- Emily Riehl, *Complicial sets, an overture*, 2016 MATRIX annals, MATRIX Book Ser., vol. 1, Springer, Cham, 2018, pp. 49–76. MR 3792516
- Dominic Verity, *Weak complicial sets. I. Basic homotopy theory*, Adv. Math. **219** (2008), no. 4, 1081–1149. MR 2450607
- Enrico M. Vitale, *Monoidal categories for Morita theory*, Cahiers Topologie Géom. Différentielle Catég. **33** (1992), no. 4, 331–343. MR 1197429