The Morita $(\infty, 2)$-Category of a Monoidal Category as a 2-Complicial Set

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We present an explicit and elementary construction of the Morita $(\infty, 2)$-category associated to a monoidal category $\mathcal{C}^{\otimes}$ satisfying minimal conditions [1]. This construction provides a self-contained approach to understanding the higher categorical structure formed by monoids, bimodules, and bimodule maps.

Our main result constructs this structure as a 3-coskeletal 2-complicial set $M^{\natural}(\mathcal{C}^{\otimes})$—a model for $(\infty, 2)$-categories based on marked simplicial sets due to Verity [2]. In this construction, vertices encode monoids, edges represent bimodules, triangles capture bimodule maps $\varphi_{012} : M_{01} \otimes_{A_1} M_{12} \to M_{02}$ from balanced tensor products, and tetrahedra encode coherence conditions of the form

$$(\varphi_{012} \otimes_{A_2} M_{23}) \bullet \varphi_{023} = \alpha_{M_{01}|M_{12}|M_{23}} \bullet (M_{01} \otimes_{A_1} \varphi_{123}) \bullet \varphi_{013}.$$

The marking distinguishes invertible structure: marked edges correspond to invertible bimodules, while marked triangles represent bimodule isomorphisms.

The key technical requirement is that $\mathcal{C}^{\otimes}$ admits a *calculus of balanced tensor products*—conditions ensuring that coequalizers of the form

$$M \otimes_B N := \operatorname{coeq}\left[(M \otimes B) \otimes N \underset{\alpha_{M,B,N} \bullet (M \otimes \ell_N)}{\overset{r_M \otimes N}{\rightrightarrows}} M \otimes N \right]$$

exist and that functors $M \otimes (-)$ and $(-) \otimes P$ preserve them appropriately. This framework encompasses important examples including $\mathbf{Ab}^{\otimes}$, $\mathbf{Set}^{\amalg}$, and more generally, cocomplete closed monoidal categories.

Rather than directly verifying the bicategory axioms (a lengthy but well-believed result), we leverage the combinatorics of simplicial sets to reformulate these coherence conditions. We prove that $M^{\natural}(\mathcal{C}^{\otimes})$ satisfies the defining properties of a 2-complicial set by establishing the appropriate lifting properties against complicial horns $\Lambda^k[m] \hookrightarrow \Delta^k[m]$, thinness extensions $\Delta^k[m]' \hookrightarrow \Delta^k[m]''$, and saturation extensions $\Delta[3]^{\mathrm{eq}} \star \Delta[\ell] \hookrightarrow \Delta[3]^{\sharp} \star \Delta[\ell]$.

This approach offers both a proof of concept for efficiently encoding higher categorical structure and a pathway toward scaling these methods to even higher dimensions, such as when treating braided or symmetric monoidal categories. Our construction recovers known structures—such as $M^{\natural}(\mathbf{Ab}^{\otimes}) \simeq N_{\mathrm{Duskin}}(\mathbf{Alg}^{\mathbf{bi}})$ [3] and $M^{\natural}(\mathbf{Set}^{\amalg}) \simeq N_{\mathrm{Duskin}}(\mathbf{Span})$—while providing an explicit, elementary foundation that avoids heavy machinery.

[1] A. Dutta, S. Luneia, M. Rovelli, and S. Silver, *The Morita $(\infty, 2)$-category of a monoidal category as a 2-complicial set*, arXiv preprint arXiv:2509.21472, 2025.

[2] Dominic Verity. Complicial Sets. arXiv preprint math/0410412, 2005. `https://arxiv.org/abs/math/0410412`.

[3] N. Gurski. Nerves of bicategories as stratified simplicial sets. *J. Pure Appl. Algebra*, 213(6):927–946, 2009.