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ON THE EXISTENCE AND UNIQUENESS OF HAAR MEASURE ON LOCALLY COMPACT TOPOLOGICAL GROUPS

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Introduction

The purpose of this project is to prove the existence and uniqueness of Haar measure on locally compact topological groups. We begin by stating a few definitions which will be used throughout the subsequent sections. In order to motivate only focusing on left Haar measure, we show that given a left Haar measure, one immediately obtains a right Haar measure. We then provide a proof of the existence of left Haar measure on a locally compact topological group. Then, after a couple of lemmas, we prove uniqueness of left Haar measure on a locally compact topological group. We note here that, by uniqueness, we mean that any two Haar measures on a locally compact topological group are not exactly the same, but in fact only differ by a positive multiplicative constant. We then briefly note how the relation between left and right Haar measure immediately also implies existence and uniqueness of right Haar measure.

Basic Definitions

To begin with, we quickly review some basic definitions and notations that we will use throughout the project.

Definition 2.0 (Topological group) A topological group $\mathscr{G}$ is a topological space equipped with a group structure along with the condition that the multiplication map $m : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ and the inverse map $(.)^{-1} : \mathscr{G} \to \mathscr{G}$ are continuous. In category theoretical language, a topological group is nothing but a group object in the category **Top**.

Definition 2.1 (Borel Subset) Let X be a topological space with topology τ and let $A \subseteq X$. Then, A is said to be a Borel subset of X iff $A \in \sigma[\tau]$, i.e, the sigma algebra generated by τ.

Definition 2.2 (Topological Measure Space) A topological measure space is a measure space $(X, \sum, \mu)$, where X is the topological space, $\sum$ is the sigma algebra of measurable subsets, and μ is the measure, such that X is a topological space and $\sum$ is exactly the collection of Borel subsets of X.

Definition 2.3 (Borel Measure) A measure μ on a topological measure space X is called a Borel measure iff X is Hausdorff.
The reason we add the extra condition of T_2 instead of doing things in complete generality, is that, first of all, most spaces we care about in practice are going to be Hausdorff anyways, and furthermore, we would like to know that compact subsets are measurable (because in Hausdorff spaces compact subsets are closed), and in general this won't necessarily be the case.

Definition 2.4 (Regular Measure) Let $(X, \sum, \mu)$ be a Borel measure space. Then, μ is said to be regular, or sometimes a regular Borel measure, iff it has the following properties-
(i) Whenever $K \subseteq X$ is compact, then $\mu(K) < \infty$.
(ii)Whenever $A \in \sum$, then, $\mu(A) = \inf\{\mu(U) \mid A \in U, U \text{ is open}\}$. This property is called the outer regularity of μ.
(iii)Whenever $A \in \sum$, then $\mu(A) = \sup\{\mu(K) \mid K \in A, K \text{ is compact}\}$. This property is called the inner

regularity of μ.

Definition 2.5 (Locally Compact space) A topological space X is called locally compact if for every point $x \in X$, there exists an open set $x \in U \subseteq X$ and a compact $K \subseteq X$ such that $U \subseteq K$. If X is Hausdorff, this is equivalent to saying that every point $x \in X$ has a compact neighbourhood.

Definition 2.6 (Locally Compact Group) A locally compact group is a topological group $\mathscr{G}$ that is locally compact and T_1.

Definition 2.7 (Haar Measure) Let $\mathscr{G}$ be a topological group. A left Haar measure (resp. right Haar measure) on $\mathscr{G}$ is a nonzero regular Borel measure μ on $\mathscr{G}$ such that $\mu(gA) = \mu(A)$ (resp. $\mu(Ag)$ $= \mu(A)$) for all $g \in \mathscr{G}$ and all measurable subsets A of $\mathscr{G}$.

The Existence

Lemma 1. *Let $\mathscr{G}$ be a topological group, let K be a compact subset of $\mathscr{G}$, and let U be an open subset of $\mathscr{G}$ such that $K \subseteq U$. Then, there is an open set V containing the identity such that $KV \subseteq U$.*

Proof. For each $x \in K$, define $W_x = x^{-1}U$. Because $x \in U$, W_x is an open neighborhood of the identity. Then, pick V_x to be an open neighborhood of the identity such that $V_xV_x \subseteq W_x$. Then, the collection $\{xV_x \mid x \in K\}$ is an open cover of K, so there is a finite collection of points $x_1, ..., x_n$ such that $K \subseteq \bigcup_{i=1}^{n} x_kV_{x_k}$. Define $V = \bigcap_{k=1}^{n} V_{x_k}$. Let $x \in K$. Then, there is some x_k such that $x \in x_kV_{x_k}$, so that $xV \subseteq x_kV_{x_k}V_{x_k} \subseteq x_kW_{x_k} = U$. Thus, $KV \subseteq U$. $\square$

Lemma 2. *Let X be a Hausdorff space, let K be a compact subset of X, and let U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then, there are compact sets K_1 and K_2 of X such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, and K= $K_1 \cup K_2$.*

Proof. Define $L_1 = K - U_1$ and $L_2 = K - U_2$. K is closed because X is Hausdorff, so each L_i is closed. Because each L_i is a closed subspace of K and K is compact, it follows that each L_i is also compact. Furthermore, because $K \subseteq U_1 \cup U_2$, $L_1 \cap L_2 = \phi$. Because L_1 and L_2 are disjoint compact subsets of a Hausdorff space, we can separate them with disjoint open sets, say V_1 and V_2 respectively. Define $K_1 = K - V_1$ and $K_2 = K - V_2$. Similarly as before, both K_1 and K_2 are compact. Now, $K_1 = K - V_1 \subseteq K - L_1 = K - (K - U_1) = K \cap (K \cap U_1^C)^C = K \cap (K^C \cup U_1) \subseteq U_1$. Similarly, $K_2 \subseteq U_2$. Furthermore, $K_1 \cup K_2 = K - (V_1 \cap V_2) = K$. $\square$

Theorem 1. *Let $\mathscr{G}$ be a locally compact topological group.Then, there exists a left Haar measure on $\mathscr{G}$.*

Proof. Let K be a compact subset of $\mathscr{G}$ and let V be a subset of $\mathscr{G}$ with nonempty interior. Then, $\{gV^\circ \mid g \in \mathscr{G}\}$ is an open cover of K, so there are a finite number of elements of $\mathscr{G}$, $g_1, ..., g_n$, such that $K \subseteq \bigcup_{k=1}^{n} g_kV^\circ$. Let $(K : V)$ denote the smallest nonnegative integer for which such a sequence exists. Now, let κ denote the collection of compact subsets of $\mathscr{G}$ and let $\mathscr{U}$ denote the collection of open subsets of $\mathscr{G}$ containing the identity. Because $\mathscr{G}$ is locally compact, there is a compact subset of $\mathscr{G}$ with nonempty interior, call it K_0. For each $U \in \mathscr{U}$, define a function $\mu_U : \kappa \to \mathbb{R}$ such that $\mu_U(K) = \frac{(K:U)}{(K_0:V)}$. Because K_0 is nonempty, $(K_0 : U) \neq 0$, and so μ_U is well-defined.
As $(K : U)$ is always a nonnegative integer, μ_U is clearly always nonnegative. We now show that $(K : U) \leq (K : K_0)(K_0 : U)$ for $K \in \kappa$ and $U \in \mathscr{U}$. For the remainder of this paragraph, let us write $m = (K : K_0)$ and $n = (K_0 : U)$. Then, let $g_1, ..., g_m \in \mathscr{G}$ and let $h_1, ..., h_n \in \mathscr{G}$ be such that $K \subseteq \bigcup_{k=1}^{m} g_kK_0^\circ$ and $K_0 \subseteq \bigcup_{k=1}^{n} h_kU$. Then, $K \subseteq \bigcup_{i=1}^{m}[\bigcup_{j=1}^{n} g_ih_jU]$, so that K can be covered by mn cosets of U, so that $(K : U) \leq mn = (K : K_0)(K_0 : U)$. It follows that $0 \leq \mu_U(K) \leq (K : K_0)$.

Now we construct the Haar measure on κ . Define $X = \prod_{K \in \kappa}[0, (K : K_0)]$. Because $0 \leq \mu_U(K) \leq (K : K_0)$, each μ_U may be thought of as a point in X. Thinking of each μ_U as a point in X, for each $V \in U$, define $C(V) = \overline{\{\mu_U \mid U \in U, U \subseteq V\}}$. We wish to show that the collection $\{C(V) \mid V \in \mathscr{U}\}$ possess the finite intersection property, so let $V_1, ..., V_n \in \mathscr{U}$. Then, $\mu_{\bigcap_{k=1}^{n} V_k} \in \bigcap_{k=1}^{n} C(V_k)$ implying, $\bigcap_{k=1}^{n} C(V_k)$ is nonempty. Thus, $\{C(V) \mid V \in \mathscr{U}\}$ satisfies the finite intersection property, and because X is compact by Tychonoff's Theorem, it follows that $\bigcap_{V \in \mathscr{U}} C(V)$ is nonempty, so we may pick some $\mu \in \bigcap_{V \in \mathscr{U}} C(V)$.

Next we show the monotonicity of μ on κ. Let $K_1, K_2 \in \kappa$ be such that $K_1 \subseteq K_2$. We first show that, for each $U \in \mathscr{U}, \mu_U(K_1) \le \mu_U(K_2)$. But this is trivial, because the covering of K_2 with $(K_2 : U)$ cosets of U is also a covering of K_1 with $(K_2 : U)$ cosets of U, so that $(K_1 : U) \le (K_2 : U)$, and hence $\mu_U(K_1) \le \mu_U(K_2)$. We think of elements f of X as functions from κ to $\mathbb{R}$, consider the map that sends $f \in X$ to $f(K_2) - f(K_1)$. This is a composition of continuous functions, and hence continuous. This map is also nonnegative on each $C(V)$ because $\mu_U(K_1) \le \mu_U(K_2)$ for each $U \in \mathscr{U}$ (we need continuity so that we know it is nonnegative on the entire closure). It follows that this map is also nonnegative at μ, so that $\mu(K_2) - \mu(K_1) \ge 0$, so that $\mu(K_1) \le \mu(K_2)$.

Next we show that μ is finitiely subadditive on κ, i.e. $\mu(K_1 \cup K_2) \le \mu(K_1) + \mu(K_2)$. Let $K_1, K_2 \in \kappa$. We first try to show that $\mu_U(K_1 \cup K_2) \le \mu_U(K_1) + \mu_U(K_2)$ for each $U \in \mathscr{U}$. Thus this is trivial, because a covering of K_1 with $(K_1 : U)$ cosets of U together with a covering of K_2 with $(K_2 : U)$ cosets of U, is a cover of $K_1 \cup K_2$ with $(K_1 : U) + (K_2 : U)$ cosets of U, so that $(K_1 \cup K_2 : U) \le (K_1 : U) + (K_2 : U)$. It follows that $\mu_U(K_1 \cup K_2) \le \mu_U(K_1) + \mu_U(K_2)$. Proceeding similarly as before, the map that sends $f \in X$ to $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is continuous and nonnegative on each $C(V)$, and hence is nonnegative for $\mu \in X$. Thus, $\mu(K_1 \cup K_2) \le \mu(K_1) + \mu(K_2)$.

Now, we would like to show that μ is finitely additive on κ. But we first show that $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ if $K_1U^{-1} \cap K_2U^{-1} = \emptyset$. Let $K_1, K_2 \in \kappa$ be such that $K_1U^{-1} \cap K_2U^{-1} = \emptyset$. Let $g_1, ..., g_n$ be such that n $= (K_1 \cup K_2 : U)$ and $K_1 \cup K_2 \subseteq \bigcup_{k=1}^n g_kU$. If some g_kU intersects both K_1 and K_2, then $g_k \in K_1U^{-1} \cap K_2U^{-1}$, a contradiction. Thus, each g_kU intersects either K_1 or K_2, but not both. Thus, we may find some natural number m with $0 \le m \le n$ and reindex the g_k's so that $K_1 \subseteq \bigcup_{k=1}^m g_kU$ and $K_2 \subseteq \bigcup_{k=m+1}^n g_kU$. Thus, $(K_1 : U) + (K_2 : U) \le (K_1 \cup K_2 : U)$. Combining this result with the previous step, it follows that $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ for each $U \in \mathscr{U}$.
Now, let $K_1, K_2 \in \kappa$ be such that $K_1 \cap K_2 = \emptyset$. Then, we may find disjoint open sets U_1 and U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$. By Lemma 1, there are open neighborhoods of the identity V_1 and V_2 such that $K_1V_1 \subseteq U_1$ and $K_2V_2 \subseteq U_2$. Define $V = V_1 \cap V_2$. Then, K_1V and K_2V are disjoint because U_1 and U_2 are disjoint. Thus, for any $U \in \mathscr{U}$ with $U \subseteq V^{-1}$, we have that $K_1U^{-1} \cap K_2U^{-1} = \emptyset$, so that, by the previous step, $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$. Thus, the continuous map from X to $\mathbb{R}$ that sends $f \in X$ to $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is 0 for each $f \in C(V^{-1})$. In particular, $\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2)$.

Our next goal is to extend μ to $2^{\mathscr{G}}$.
For $U \subseteq \mathscr{G}$ open, we define $\mu(U) = \sup\ \{\mu(K) \mid K \subseteq U, K \in \kappa\}$. We must show that if K is compact and open, these two definitions of $\mu(K)$ agree. That is, we must show that $\mu(K) = \sup\{\mu(K') \mid K' \subseteq K, K' \in \kappa\}$, where here the LHS is the original definition of μ as a point in $\bigcap_{U \in \mathscr{U}} C(U)$. Trivially, since $\mu(K) \in \{\mu(K') \mid K' \subseteq K, K' \in \kappa\}$, $\mu(K) \le \sup\ \{\mu(K') \mid K' \subseteq K, K' \in \kappa\}$. On the other hand, by the monotonicity of μ in κ, the set $\{\mu(K') \subseteq K, K' \in \kappa\}$ is bounded above by $\mu(K)$, so that, $\sup\{\mu(K') \mid K' \subseteq K, K' \in \kappa\} \le \mu(K)$. Thus, this definition agrees with the previous one. It follows trivially that this extension still satisfies the property $\mu(U_1) \le \mu(U_2)$ if $U_1 \subseteq U_2$. Now, for an arbitrary subset A of $\mathscr{G}$, define $\mu(A) = \inf\{\mu(U) \mid A \subseteq U, U \text{ is open in } \mathscr{G}\}$. Similarly as before, this indeed is an extension of our previous definition of μ to all subsets of $\mathscr{G}$. It again follows trivially that this extension still satisfies the property that $\mu(A_1) \le \mu(A_2)$ if $A_1 \subseteq A_2$. Okay, so now we will show that μ is an outer measure on $\mathscr{G}$.
Firstly, since $(\emptyset : U) = 0$ for every $U \in \mathscr{U}$, $\mu(\emptyset) = 0$. Furthermore, to show that μ is nonnegative, because of the definitions of the extensions, it suffices to show that μ is nonnegative on κ. For a fixed $K \in \kappa$, the map that sends $f \in X$ to $f(K)$ is continuous by similar reasoning as before. Furthermore, because this map is nonnegative at each μ_U , it follows that this map is nonnegative on each $C(V)$. Thus, this map is nonnegative at μ, so that $\mu(K) \ge 0$. Now, we show that μ is countably subadditive. We do it in two steps.
We first show that for each countable collection of open sets in $\mathscr{G}$, $\{U_n \mid n \in \mathbb{N}\}$, we have that $\mu(\bigcup_{n\in\mathbf{N}} U_n) \le \sum_{n\in\mathbf{N}} \mu(U_n)$. Let $\{U_n \mid n \in \mathbb{N}\}$ be a countable collection of open subsets of $\mathscr{G}$. Let K be a compact subset of $\bigcup_{n\in\mathbb{N}} U_n$. Then, $K \subseteq \bigcup_{k=1}^n U_k$ for some $n \in \mathbb{N}$. Now, by Lemma 2, we find compact sets $K_1, ..., K_n$ such that $K = \bigcup_{k=1}^n K_k$ and $K_k \subseteq U_k$ for $1 \le$ k $\le$ n. Then, inductively, $\mu(K) \le \sum_{k=1}^n \mu(K_k) \le \sum_{k=1}^n \mu(U_K) \le \sum_{n\in\mathbf{N}} \mu(U_n)$. It follows that, $\mu(\bigcup_{n\in\mathbf{N}} U_n) = \sup\{\mu(K) \mid K \subset \bigcup_{n\in\mathbf{N}} U_n, K \in \kappa\} \le \sum_{n\in\mathbf{N}} \mu(U_n)$. Now, let $\{A_n \mid n \in \mathbb{N}\}$ be an arbitrary countable collection of subsets of $\mathscr{G}$. If $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$ then trivially $\mu(\bigcup_{n\in\mathbb{N}} A_n) \le \sum_{n\in\mathbb{N}} \mu(A_n)$, so suppose, $\sum_{n\in\mathbf{N}} \mu(A_n) < \infty$. Let $\epsilon > 0$, and for each $n \in \mathbb{N}$, pick an open set U_n such that $A_n \subseteq U_n$ and $\mu(U_n) \le \mu(A_n) + \frac{\epsilon}{2n}$. Then, $\mu(\bigcup_{n\in\mathbb{N}} A_n) \le \mu(\bigcup_{n\in\mathbf{N}} U_n) \le \sum_{n\in\mathbb{N}} \mu(U_n) \le \sum_{n\in\mathbb{N}} \mu(A_n) + \epsilon \sum_{n\in\mathbb{N}} \frac{1}{2^n} = \sum_{n\in\mathbb{N}} \mu(A_n) + \frac{\epsilon}{2}$, but since

$\epsilon > 0$ was arbitrary, we have that $\mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$, which proves that μ is an outer measure.

Next we show that the collection of Caratheodory measurable sets contain the Borel subsets of $\mathscr{G}$.
To show that the collection of Caratheodory measurable sets contain the Borel subsets of $\mathscr{G}$, it suffices to show that every open subset of $\mathscr{G}$ is measurable since the collection of measurable sets form a σ-algebra, if this collection contains the topology τ of $\mathscr{G}$, then it certainly contains the σ-algebra generated by the topology τ. So let $U \subseteq \mathscr{G}$ be open and let $A \subseteq \mathscr{G}$. If $\mu(A) = \infty$, then trivially $\mu(A) \geq \mu(A \cap U) + \mu(A \cap U^C)$, so we might as well assume that $\mu(A) < \infty$. Let $\epsilon > 0$ and pick $V \subseteq \mathscr{G}$ open and such that $A \subseteq V$ and $\mu(V) \leq \mu(A) + \epsilon$. Let K be a compact subset of $V \cap U$ such that $\mu(V \cap U) - \epsilon \leq \mu(K)$, and let L be a compact subset of $V \cap K^C$ such that $\mu(V \cap K^C) - \epsilon \leq \mu(L)$. Since $K \subseteq U, V \cap U^C \subseteq V \cap K^C$, so $\mu(V \cap U^C) - \epsilon \leq \mu(V \cap K^C) - \epsilon \leq \mu(L)$. Therefore, $\mu(A \cap U) + \mu(A \cap U^C) - 2\epsilon \leq \mu(V \cap U) + \mu(V \cap U^C) - 2\epsilon \leq \mu(K) + \mu(L) = \mu(K \cup L) \leq \mu((V \cap U) \cup (V \cap K^C) \leq \mu(V) \leq \mu(A) + \epsilon$. It follows that $\mu(A \cap U) + \mu(A \cap U^C) \leq \mu(A) + 3\epsilon$. Since ϵ is arbitrary, we have that $\mu(A \cap U) + \mu(A \cap U^C) \leq \mu(A)$, and hence U is measurable. It follows that μ restricts to a measure on the Borel subsets of $\mathscr{G}$, so that it is a Borel measure ($\mathscr{G}$ is completely regular, as it is locally compact Hausdorff).
Moreover, $\mu_U(K_0) = 1$ for each $U \in \mathscr{U}$, and the continuous function that maps $f \in X$ to $f(K_0)$ is a constant 1 on each $C(U)$, and in particular $\mu(K_0) = 1$, and hence μ is nonzero. Also, considering μ as an element of X, μ is finite on compact sets. Furthermore, as by construction $\mu(A) = \inf \{\mu(U) \mid A \subseteq U, U$ is open$\}$, μ is trivially outer regular. Similarly, μ is trivially inner regular. Hence, we showed that the extension agreed with its definition for open sets which is by construction inner regular.
We are almost there, it remains to show thta μ is left translation invariant by the elements of $\mathscr{G}$.
Fix $g \in \mathscr{G}$. The elements $x_1, ..., x_n$ generate a cover for K iff the elements $gx_1, ..., gx_n$ generate a cover of gK, so that $(K : U) = (gK : U)$ for each $U \in \mathscr{U}$, and hence $\mu_U(K) = \mu_U(gK)$ for each $U \in \mathscr{U}$. It follows that the continuous function that maps $f \in X$ to $f(K) - f(gK)$ is 0 on each $C(U)$, and hence $\mu(K) = \mu(gK)$. Thus, μ is left translation invariant, and hence a left Haar measure on $\mathscr{G}$. □

Proposition 1. *Let $\mathscr{G}$ be a topological group, let μ be a Haar measure on $\mathscr{G}$, and define $\mu'(A) = \mu(A^{-1})$. Then, μ' is a right Haar measure if μ is a left Haar measure on $\mathscr{G}$.*

Proof. Suppose that μ is a left Haar measure on $\mathscr{G}$. We first show that μ' is a Borel measure on $\mathscr{G}$. Firstly, since $(.)^{-1} : \mathscr{G} \to \mathscr{G}$ is a homeomorphism, it is easy to see that A is a Borel subset of $\mathscr{G}$ iff A^{-1} is a Borel subset of $\mathscr{G}$. So, μ' is well defined and trivially, μ' is nonnegative and $\mu'(\emptyset) = 0$. Let $\{A_n \mid n \in \mathbb{N}\}$ be a countable collection of pairwise disjoint measurable subsets of $\mathscr{G}$, so $\{A_n^{-1} \mid n \in \mathbb{N}\}$ is also a countable collection of pairwise disjoint measurable subsets. Hence, $\mu'(\bigcup_{n\in\mathbf{N}} A_n) = \mu((\bigcup_{n\in\mathbb{N}} A_n)^{-1}) = \mu(\bigcup_{n\in\mathbb{N}} A_n^{-1}) = \sum_{n\in\mathbb{N}} \mu(A_n^{-1}) = \sum_{n\in\mathbb{N}} \mu'(A_n)$. Thus, μ' is a Borel measure on $\mathscr{G}$.
Let $K \subseteq \mathscr{G}$ be compact. Then, K^{-1} is also compact, so $\mu'(K) = \mu(K^{-1}) < \infty$. Let A be a measurable subset of $\mathscr{G}$, $A^{-1} \subseteq U$ and U is open iff $A \subseteq U^{-1}$ and U^{-1} is open, so that $\{\mu(U) \mid A^{-1} \subseteq U, U$ is open$\}$=$\{\mu(U^{-1}) \mid A \subseteq U, U$ is open$\}$. Then, $\mu'(A)$=inf $\{\mu(U) \mid A^{-1} \subseteq U, U$ is open$\}$=inf$\{\mu(U^{-1}) \mid A \subseteq U, U$ is open$\}$=inf$\{\mu'(U) \mid A \subseteq U, U$ is open$\}$. Similarly for A open, $\mu'(A)$=sup$\{\mu'(K) \mid K \subseteq K, K$ is compact$\}$ and hence, μ' is regular.
Finally, since μ is nonzero, μ' is nonzero as well. Also, $\mu'(Ag) = \mu((Ag)^{-1}) = \mu(g^{-1}A^{-1}) = \mu(A^{-1}) = \mu'(A)$. Therefore, μ' is a right Haar measure of $\mathscr{G}$.
This proposition tells us that, while left and right Haar measure on a group may be different, they are related in a simple manner, and so we may as well simply concern ourselves with the study of one or the other. □

The Uniqueness

Lemma 3. *Let $\mathscr{G}$ be a locally compact topological group and let $f \in C_c(\mathscr{G})$. Then, for every $\epsilon > 0$, there is an open neighborhood U of the identity such that whenever $y \in xU$, it follows that $\mid f(x) - f(y) \mid < \epsilon$.*

Proof. Define $K = supp[f]$. Let $\epsilon > 0$. By continuity of f, for each $x \in K$, we may find an open neighborhood U_x of the identity such that whenever $y \in xU_x$, it follows that $\mid f(y) - f(x) \mid < \epsilon$. Then, for each $x \in K$, choose another open neighborhood of the identity V_x such that $V_x V_x \subseteq U_x$. By compactness of K, there is a finite number of $x_1, ..., x_n$ such that $K \subseteq \cup_{k=1}^{n} x_k V_{x_k}$. Define $V = \cap_{k=1}^{n} V_{x_k}$ and define $U = V \cap V^{-1}$. U is clearly an open neighborhood of the identity, and we claim that this might be our required neighbourhood. Let $y \in xU$. If $x, y \notin K$, then $\mid f(x) - f(y) \mid = 0$, and so there is nothing to worry about, so we may assume that either $x \in K$ or $y \in K$. First suppose that $x \in K$. Because $x \in K$, it follows that $x \in x_k V_{x_k}$ for some $1 \leq$ k $\leq$ n, and hence that $x \in x_k U_{x_k}$. On the other hand, because $x \in x_k V_{x_k}$ and $V \subseteq V_{x_k}$, it follows that $y \in xV \subseteq x_k V_{x_k} V_{x_k} \subseteq x_k U_{x_k}$. Thus, $\mid f(x) - f(y) \mid \leq \mid f(x) - f(x_k) \mid + \mid f(x_k) - f(y) \mid < 2\epsilon$. Now let us suppose that $y \in K$. Now, $y = xu$ for some $u \in U$, so $x = yu^{-1}$. But $U = V \cap V^{-1}$, so $u^{-1} \in U$, so that $x \in yU$. Then, we have that $y \in K$ and $x \in yU$, so we may give similar argument as in the previous paragraph by interchanging the roles of x and y. □

Lemma 4. *Let $(X, \sum, \mu)$ be a measure space, let $f : X \to \mathbb{R}$ be measurable, and let $A \subseteq X$ be measurable. Then, if $A = \{x \in X \mid f(x) > 0\}$ and $\mu(A) > 0$, there is some $a > 0$ such that $\mu(\{x \in A \mid f(x) \geq a\}) > 0$.*

Proof. Suppose $A = \{x \in X \mid f(x) > 0\}$ and $\mu(A) > 0$. Suppose that, for all $a > 0$, $\mu(x \in A \mid f(x) \geq a)$ $= 0$. We denote, $S_n = \{x \in A \mid f(x) \geq \frac{1}{2n}\}$. Then, $A = \cup_{n \in \mathbb{N}} S_n$, so $\mu(A) \leqslant \sum_{n \in \mathbb{N}} = 0$, implying $\mu(A)$ $= 0$, a contradiction. Thus, there is some $a > 0$ such that $\mu(\{x \in A \mid f(x) \geq a\}) > 0$. □

Theorem 2. *Let $\mathscr{G}$ be a locally compact topological group, and let μ and μ' be two left Haar measures on $\mathscr{G}$. Then, $\mu = a\mu'$ for some $a > 0$, $a \in \mathbb{R}$.*

Proof. Firstly, because μ is nonzero, there is some set of nonzero measure. It follows from the outer regularity of μ that there is some open set, containing this set that is also of positive measure, and by the inner regularity of μ, it follows that there is a compact set of nonzero measure contained in this open set, we call it K. Now, let $f \in C_c(\mathscr{G})$ be nonnegative and non-vanishing. Define $U = f^{-1}(\mathbb{R}_{>0})$. U is nonempty because f is not identically 0. By continuity, U is open, so because K is compact and U is nonempty, there is a finite number of elements $g_1, ..., g_n$ in $\mathscr{G}$ such that $K \subseteq \cup_{k=1}^{n} g_k U$, so that $0 < \mu(K) \leq \sum_{k=1}^{n} \mu(g_k U) = n\mu(U)$, so that $\mu(U) > 0$. Then, by Lemma 4, it follows that there is some $a > 0$ such that, $V = \{g \in G \mid f(g) \geq a\}$ is of positive measure. It follows that $\int_G f d\mu \geqslant \int_V f d\mu \geqslant a\mu(V) > 0$.
Let $g \in C_c(\mathscr{G})$ be nonnegative and non-vanishing, and let $f \in C_c(\mathscr{G})$ be arbitrary. g will remain the same throughout the remainder of the proof. Define, $h(x, y) = \frac{f(x)g(yx)}{\int_{\mathscr{G}} g(tx) d\mu'(t)}$, clearly the denominator never vanishes, and so h is well-defined on all of $\mathscr{G} \times \mathscr{G}$. Trivially, h is compactly supported because both f and g are compactly supported. Now, to show that h is continuous, it suffices to show that $I(x) \equiv \int_{\mathscr{G}} g(tx) d\mu'(t)$ is a continuous function. Define $K = supp[g]$, let $x_0 \in \mathscr{G}$, and let U be an open neighborhood of x_0 whose closure is compact which exists because $\mathscr{G}$ is locally compact and Hausdorff. $K \times \overline{U}^{-1}$ is compact by Tychonoff's Theorem, so $K\overline{U}^{-1}$ is compact because this is the image of $K \times \overline{U}^{-1}$ under a continuous function. Let $\epsilon > 0$, and choose $\delta > 0$, so that $\delta\mu'(K\overline{U}^{-1}) < \epsilon$, which we may do because $K\overline{U}^{-1}$ is compact, and hence of finite measure. By Lemma 3, there is an open neighborhood V of the identity such that whenever $y \in xV$, it follows that $\mid g(x) - g(y) \mid < \delta$. Then, whenever $x \in U \cap x_0 V$, an open neighborhood of x_0, $tx \in tx_0 V$, so that, $\mid I(x) - I(x_0) \mid \leq \int_{\mathscr{G}} \mid g(tx) - g(tx_0) \mid d\mu'(t) \leq \delta\mu'(K\overline{U}^{-1}) < \epsilon$, where we have used the fact that integrand vanishes for t outside of $K\overline{U}^{-1}$. Thus, I is continuous, and hence h is continuous, and hence $h \in C_c(\mathscr{G} \times \mathscr{G})$. By a generalization of Fubini's Theorem, we have that,

$$\begin{aligned}
\int_{\mathscr{G}} \left[\int_{\mathscr{G}} h(x, y) d\mu'(y) \right] d\mu(x) &= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} h(x, y) d\mu(x) \right] d\mu'(y) \\
&= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} h(y^{-1}x, y) d\mu(x) \right] d\mu'(y) \\
&= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} h(y^{-1}x, y) d\mu'(y) \right] d\mu(x) \\
&= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} h(y^{-1}, xy) d\mu'(y) \right] d\mu(x).
\end{aligned}$$

$$
\begin{aligned}
\text{Thus, } \int_{\mathscr{G}} f(x)d\mu(x) &= \int_{\mathscr{G}} \left[f(x) \frac{\int_{\mathscr{G}} g(yx)d\mu'(y)}{\int_{\mathscr{G}} g(tx)d\mu'(t)} \right] d\mu(x) \\
&= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} \frac{f(x)g(yx)}{\int_{\mathscr{G}} g(tx)d\mu'(t)} d\mu'(y) \right] d\mu(x) \\
&= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} h(x,y)d\mu'(y) \right] d\mu(x) \\
&= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} h(y^{-1},xy)d\mu'(y) \right] d\mu(x) \\
&= \int_{\mathscr{G}} \left[\int_{\mathscr{G}} \frac{f(y^{-1})g(x)}{\int_{\mathscr{G}} g(ty^{-1})dt} d\mu'(y) \right] d\mu(x) \\
&= (\int_{\mathscr{G}} g(x)d\mu(x))(\int_{\mathscr{G}} \frac{f(y^{-1})}{\int_{\mathscr{G}} g(ty^{-1})d\mu'(t)} d\mu'(y))
\end{aligned}
$$

Thus, $\frac{\int_{\mathscr{G}} f(x)d\mu(x)}{\int_{\mathscr{G}} g(x)d\mu(x)} = \mathrm{C}$, where C is some constant independent of μ.
Now, since this constant does not depend on μ, we must have, $\frac{\int_{\mathscr{G}} fd\mu}{\int_{\mathscr{G}} gd\mu} = \mathrm{C} = \frac{\int_{\mathscr{G}} fd\mu'}{\int_{\mathscr{G}} gd\mu'}$ and hence, $\int_{\mathscr{G}} fd\mu'$ $= a\int_{\mathscr{G}} fd\mu$, where $a = \frac{\int_{\mathscr{G}} gd\mu'}{\int_{\mathscr{G}} gd\mu}$. Finally, for $f \in C_c(\mathscr{G})$, define, $\phi(f) = \int_{\mathscr{G}} fd\mu$ and $\psi(f) = \int_{\mathscr{G}} fd\nu$, where ν is a measure defined by $\nu = \frac{1}{a\mu'}$. Both ϕ and ψ are positive linear functions on $C_c(\mathscr{G})$, and $\phi(f) = \int_{\mathscr{G}} fd\mu = \frac{1}{a}\int_{\mathscr{G}} fd\mu' = \int_{\mathscr{G}} fd\nu = \psi(f)$. Thus, Thus, by the Riesz Representation Theorem, it follows that $\mu = \nu$, i.e. $\mu' = a\mu$ with $a \in \mathbb{R}$, $a > 0$. □

This theorem tells us that left Haar measure on $\mathscr{G}$ is essentially unique, in the sense that any two left Haar measures differ only by a positive multiplicative constant.

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