Simplicial Homotopy Type Theory

An introduction to the Category Theory of Segal types

Arghan Dutta
Advisor: Professor Siddhartha Gadgil

Indian Institute of Science

11th December 2023

Contents

1,2,3,...,∞-category

- How to think about an ∞-category?

1,2,3,...,∞-category

- How to think about an ∞-category?

In addition to the objects and arrows in an ordinary category, which we think of 0 and 1-dimensional, we have arrows between arrows, which we think of as inhabiting the 2-dimensional space. Then we have arrows between, arrows between arrows inhabiting the 3-dimensional space and so on.

1,2,3,...,∞-category

- How to think about an ∞-category?

In addition to the objects and arrows in an ordinary category, which we think of 0 and 1-dimensional, we have arrows between arrows, which we think of as inhabiting the 2-dimensional space. Then we have arrows between, arrows between arrows inhabiting the 3-dimensional space and so on.

- What is the fundamental challenge with such an object?

1,2,3,...,∞-category

- How to think about an ∞-category?

In addition to the objects and arrows in an ordinary category, which we think of 0 and 1-dimensional, we have arrows between arrows, which we think of as inhabiting the 2-dimensional space. Then we have arrows between, arrows between arrows inhabiting the 3-dimensional space and so on.

- What is the fundamental challenge with such an object?

The challenge in defining ∞-categories has to do with giving a precise mathematical meaning of the notion of a weak composition law, not just for the 1-morphisms but also for the morphisms in higher dimensions. While proving theorems about ∞-categories, we need to first pick a specific definition, like choosing coordinates and prove theorems with reference to that definition, thereby providing a translation problem.

1,2,3,...,∞-category

Few examples of models of ∞-categories are: simplicial categories, quasi-categories, Segal categories, complete Segal spaces.

1,2,3,...,∞-category

Few examples of models of ∞-categories are: simplicial categories, quasi-categories, Segal categories, complete Segal spaces.

- Is there any way out of this compatibility issue?

1,2,3,...,∞-category

Few examples of models of ∞-categories are: simplicial categories, quasi-categories, Segal categories, complete Segal spaces.

- Is there any way out of this compatibility issue?

This is a very recent approach by Riehl and Verity where an ∞-category is thought of living in their own category, the category of all ∞-categories, the ∞-cosmos.

1,2,3,...,∞-category

Few examples of models of ∞-categories are: simplicial categories, quasi-categories, Segal categories, complete Segal spaces.

- Is there any way out of this compatibility issue?

This is a very recent approach by Riehl and Verity where an ∞-category is thought of living in their own category, the category of all ∞-categories, the ∞-cosmos. The strategy is to prove theorems about ∞-cosmos in general and then they specialize to give the theorems about ∞-categories proven in absolutely every model, that apply universally.

1,2,3,...,∞-category

Few examples of models of ∞-categories are: simplicial categories, quasi-categories, Segal categories, complete Segal spaces.

- Is there any way out of this compatibility issue?

This is a very recent approach by Riehl and Verity where an ∞-category is thought of living in their own category, the category of all ∞-categories, the ∞-cosmos. The strategy is to prove theorems about ∞-cosmos in general and then they specialize to give the theorems about ∞-categories proven in absolutely every model, that apply universally.

Our goal is to give a model independent definition of an ∞-category by extending the theory obtained by adjoining bunch of new axioms and techniques to type theory, called homotopy type theory (HoTT).

HoTT = Homotopy theory + Type theory

- **What is Homotopy Type Theory (HoTT)?**

HoTT = Homotopy theory + Type theory

- **What is Homotopy Type Theory (HoTT)?**

Homotopy type theory (HoTT) is a new subject that augments Martin-Löf intensional dependent type theory (MLTT) with additional rules and axioms enabling it to be used as a formal language for reasoning about homotopy theory. HoTT provides a synthetic framework that is suitable for developing the theory of mathematical objects with natively homotopical content.

HoTT = Homotopy theory + Type theory

- **What is Homotopy Type Theory (HoTT)?**

Homotopy type theory (HoTT) is a new subject that augments Martin-Löf intensional dependent type theory (MLTT) with additional rules and axioms enabling it to be used as a formal language for reasoning about homotopy theory. HoTT provides a synthetic framework that is suitable for developing the theory of mathematical objects with natively homotopical content.

- **But, where does this homotopy perspective come from?**

HoTT = Homotopy theory + Type theory

- **What is Homotopy Type Theory (HoTT)?**

Homotopy type theory (HoTT) is a new subject that augments Martin-Löf intensional dependent type theory (MLTT) with additional rules and axioms enabling it to be used as a formal language for reasoning about homotopy theory. HoTT provides a synthetic framework that is suitable for developing the theory of mathematical objects with natively homotopical content.

- **But, where does this homotopy perspective come from?**

In HoTT, we regard the types as spaces (as studied in homotopy theory) or higher groupoids (as studied in higher category theory), and the logical constructions (such as the product $A \times B$) as homotopy-invariant contructions on these spaces.

Where is Homotopy in HoTT?

To briefly explain this perspective, consider first the basic concept of type theory, namely that the term a is of type A, which is written:

Where is Homotopy in HoTT?

To briefly explain this perspective, consider first the basic concept of type theory, namely that the term a is of type A, which is written:

$$a : A$$

Where is Homotopy in HoTT?

To briefly explain this perspective, consider first the basic concept of type theory, namely that the term a is of type A, which is written:

$$a : A$$

In HoTT, we think of it as:

Where is Homotopy in HoTT?

To briefly explain this perspective, consider first the basic concept of type theory, namely that the term a is of type A, which is written:

$$a : A$$

In HoTT, we think of it as:

"a is a point of the space A"

Where is Homotopy in HoTT?

To briefly explain this perspective, consider first the basic concept of type theory, namely that the term a is of type A, which is written:

$$a : A$$

In HoTT, we think of it as:

"a is a point of the space A"

Similarly, every function $f : A \to B$ in type theory is regarded as a continuous map from the space A to the space B.

Where is Homotopy in HoTT?

To briefly explain this perspective, consider first the basic concept of type theory, namely that the term a is of type A, which is written:

$$a : A$$

In HoTT, we think of it as:

"a is a point of the space A"

Similarly, every function $f : A \to B$ in type theory is regarded as a continuous map from the space A to the space B.

Now, the key idea of the homotopy interpretation is that the logical notion of equality, identity type $a =_A b$ of two terms $a, b : A$ of the same type A can be understood as the existence of a path $p : a \rightsquigarrow b$ from point a to point b in the space A.

Where is Homotopy in HoTT?

To briefly explain this perspective, consider first the basic concept of type theory, namely that the term a is of type A, which is written:

$$a : A$$

In HoTT, we think of it as:

"a is a point of the space A"

Similarly, every function $f : A \to B$ in type theory is regarded as a continuous map from the space A to the space B.

Now, the key idea of the homotopy interpretation is that the logical notion of equality, identity type $a =_A b$ of two terms $a, b : A$ of the same type A can be understood as the existence of a path $p : a \rightsquigarrow b$ from point a to point b in the space A. This also means that two functions $f, g : A \to B$ can be identified if they are homotopic, since a homotopy is just a (continuous) family of paths $p_x : f(x) \rightsquigarrow g(x)$, one for each $x : A$.

Types as groupoids

The first natural, non-trivial semantics for intensional type theory were developed by Hofmann and Streicher in their disproof of the uniqueness of identity proofs (UIP) using groupoids.

Types as groupoids

The first natural, non-trivial semantics for intensional type theory were developed by Hofmann and Streicher in their disproof of the uniqueness of identity proofs (UIP) using groupoids. Intuitively, the identity type over a groupoid G is interpreted as the discrete groupoid $Arr(G)$ of arrows in G, so that an identity witness $f : a =_A b$ becomes an arrow $f : a \to b$ in G.

Types as groupoids

The first natural, non-trivial semantics for intensional type theory were developed by Hofmann and Streicher in their disproof of the uniqueness of identity proofs (UIP) using groupoids. Intuitively, the identity type over a groupoid G is interpreted as the discrete groupoid $Arr(G)$ of arrows in G, so that an identity witness $f : a =_A b$ becomes an arrow $f : a \to b$ in G.

However, unlike in type theory, these cannot in turn be further related by identity terms of higher type $f =_{(a =_A b)} g$, since a (conventional) groupoid generally has no such higher-dimensional structure.

Types as groupoids

The first natural, non-trivial semantics for intensional type theory were developed by Hofmann and Streicher in their disproof of the uniqueness of identity proofs (UIP) using groupoids. Intuitively, the identity type over a groupoid G is interpreted as the discrete groupoid $Arr(G)$ of arrows in G, so that an identity witness $f : a =_A b$ becomes an arrow $f : a \to b$ in G.

However, unlike in type theory, these cannot in turn be further related by identity terms of higher type $f =_{(a =_A b)} g$, since a (conventional) groupoid generally has no such higher-dimensional structure.

Thus the groupoid semantics validates a certain truncation principle, stating that all higher identity types are trivial, a form of extensionality one dimension up. The groupoid laws for the identity types are strictly satisfied in these models, rather than holding only up to propositional equality.

Types as ∞-groupoids

This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types.

Types as ∞-groupoids

This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types.

Such structures indeed occurs naturally , we recall that every topological space has a fundamental ∞-groupoid, Π_∞.

Types as ∞-groupoids

This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types.

Such structures indeed occurs naturally , we recall that every topological space has a fundamental ∞-groupoid, Π_∞.

- **How can we have a similar semantics for a type in HoTT?**

Types as ∞-groupoids

This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types.

Such structures indeed occurs naturally , we recall that every topological space has a fundamental ∞-groupoid, Π_∞.

- **How can we have a similar semantics for a type in HoTT?**

In HoTT, each type A can be seen to have the structure of an ∞-groupoid.

Types as ∞-groupoids

This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types.

Such structures indeed occurs naturally , we recall that every topological space has a fundamental ∞-groupoid, Π_∞.

- **How can we have a similar semantics for a type in HoTT?**

In HoTT, each type A can be seen to have the structure of an ∞-groupoid. We can iterate the identity type: we can form the type $p =_{(x=_A y)} q$ of identifications between identifications p, q, and the type $r =_{(p=_{(x=_A y)} q)} s$, and so on.

Types as ∞-groupoids

This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types.

Such structures indeed occurs naturally , we recall that every topological space has a fundamental ∞-groupoid, Π_∞.

- **How can we have a similar semantics for a type in HoTT?**

In HoTT, each type A can be seen to have the structure of an ∞-groupoid. We can iterate the identity type: we can form the type $p =_{(x =_A y)} q$ of identifications between identifications p, q, and the type $r =_{(p =_{(x =_A y)} q)} s$, and so on. The structure of this tower of identity types corresponds precisely to that of the continuous paths and higher dimensional homotopies between them in a space, or an ∞-groupoid.

HoTT as a synthetic foundation

- **Why is HoTT synthetic in nature?**

HoTT as a synthetic foundation

- **Why is HoTT synthetic in nature?**

Unlike in classical homotopy theory, where one studies spaces analytically, i.e, with the help of a topology, simplicial sets, or any other combinatorial gadget, in HoTT the notions of points, paths, and paths between paths are basic, indivisible and primitive.

HoTT as a synthetic foundation

- **Why is HoTT synthetic in nature?**

Unlike in classical homotopy theory, where one studies spaces analytically, i.e, with the help of a topology, simplicial sets, or any other combinatorial gadget, in HoTT the notions of points, paths, and paths between paths are basic, indivisible and primitive.

Thus, HoTT can be viewed as a synthetic theory of ∞-groupoids.

HoTT as a synthetic foundation

- **Why is HoTT synthetic in nature?**

Unlike in classical homotopy theory, where one studies spaces analytically, i.e, with the help of a topology, simplicial sets, or any other combinatorial gadget, in HoTT the notions of points, paths, and paths between paths are basic, indivisible and primitive.

Thus, HoTT can be viewed as a synthetic theory of ∞-groupoids.

As a result, HoTT serves as a presentation of ∞-groupoids in different homotopy theoretic models. One such classic model is the Voevodsky's simplicial set model in which types are regarded as Kan complexes in the Quillen model structure. In this interpretation, the identity type $a =_K b$ of any two points a, b in a Kan complex K is itself a Kan complex.

Homotopical trinitarianism?

$$
\begin{tikzcd}
 & \text{Homotopical mathematics} \arrow[dl, "\text{formalize}"'] \arrow[dr, "\text{internalize}"] & \\
\text{HoTT} \arrow[rr, "\text{interpret}"'] & & \text{Higher categories}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \text{Homotopical mathematics} \arrow[dr, "\text{category of structures}"] & \\
\text{HoTT} \arrow[ur, "\text{synthetic theory}"] & & \text{Higher categories} \arrow[ll, "\text{internal logic}"]
\end{tikzcd}
$$

Aim

- **Can HoTT be viewed as a tool to synthetically study higher categories in general?**

Aim

• **Can HoTT be viewed as a tool to synthetically study higher categories in general?**

Riehl and Shulman introduced a new type theory by adding to HoTT a strict interval object and the novel idea, due to Shulman and Lumsdaine, of a new type former called extension type. We call this simplicial homotopy type theory (sHoTT).

Aim

• **Can HoTT be viewed as a tool to synthetically study higher categories in general?**

Riehl and Shulman introduced a new type theory by adding to HoTT a strict interval object and the novel idea, due to Shulman and Lumsdaine, of a new type former called extension type. We call this simplicial homotopy type theory (sHoTT).

We interpret HoTT in the Reedy model structure on bisimplicial sets and internally identify some types that correspond to some bisimplicial sets of interest.

Aim

- **Can HoTT be viewed as a tool to synthetically study higher categories in general?**

Riehl and Shulman introduced a new type theory by adding to HoTT a strict interval object and the novel idea, due to Shulman and Lumsdaine, of a new type former called extension type. We call this simplicial homotopy type theory (sHoTT).

We interpret HoTT in the Reedy model structure on bisimplicial sets and internally identify some types that correspond to some bisimplicial sets of interest.

$$\begin{tikzcd}
& \text{Types} \arrow[d, leftrightarrow] & & \text{Types with Composition and Univalence} \arrow[d, leftrightarrow] \\
\mathrm{Set}^{\Delta^{op}\times\Delta^{op}} \arrow[d, leftrightarrow] & \mathit{Reedy} \arrow[l, hook'] & \mathit{Segal} \arrow[l, hook'] \arrow[d, leftrightarrow] & \mathit{Rezk} \arrow[l, hook'] \\
\text{Bisimplicial Sets} & & \text{Types with Composition} &
\end{tikzcd}$$

Reedy model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$

Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ be a model category, then the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ has a model structure in which a map $A \to B$ is

Reedy model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$

Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ be a model category, then the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ has a model structure in which a map $A \to B$ is

- A weak equivalence iff $A_x \to B_x$ is a weak equivalence in $\mathcal{M}$ for all $x \in \mathcal{C}$.

Reedy model structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$

Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ be a model category, then the functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ has a model structure in which a map $A \to B$ is

- A weak equivalence iff $A_x \to B_x$ is a weak equivalence in $\mathcal{M}$ for all $x \in \mathcal{C}$.
- A cofibration iff the induced map $A_x \sqcup_{L_x A} L_x B \to B_x$ is a cofibration in $\mathcal{M}$ for all $x \in \mathcal{C}$

Reedy model structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$

Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ be a model category, then the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ has a model structure in which a map $A \to B$ is

- A weak equivalence iff $A_x \to B_x$ is a weak equivalence in $\mathcal{M}$ for all $x \in \mathcal{C}$.
- A cofibration iff the induced map $A_x \sqcup_{L_x A} L_x B \to B_x$ is a cofibration in $\mathcal{M}$ for all $x \in \mathcal{C}$
- A fibration iff the induced map $A_x \to B_x \times_{M_x B} M_x A$ is a fibration in $\mathcal{M}$ for all $x \in \mathcal{C}$.

Reedy model structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$

Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ be a model category, then the functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ has a model structure in which a map $A \to B$ is

- A weak equivalence iff $A_x \to B_x$ is a weak equivalence in $\mathcal{M}$ for all $x \in \mathcal{C}$.
- A cofibration iff the induced map $A_x \sqcup_{L_x A} L_x B \to B_x$ is a cofibration in $\mathcal{M}$ for all $x \in \mathcal{C}$
- A fibration iff the induced map $A_x \to B_x \times_{M_x B} M_x A$ is a fibration in $\mathcal{M}$ for all $x \in \mathcal{C}$.

We have the external product bifunctor $\mathsf{sSet} \times \mathsf{sSet} \xrightarrow{\square} \mathsf{ssSet}$, defined as $(A \square B)_{m,n} := A_m \times B_n$ which is biclosed. In particular, we have the adjoint pair

Reedy model structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$

Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ be a model category, then the functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{M})$ has a model structure in which a map $A \to B$ is

- A weak equivalence iff $A_x \to B_x$ is a weak equivalence in $\mathcal{M}$ for all $x \in \mathcal{C}$.
- A cofibration iff the induced map $A_x \sqcup_{L_x A} L_x B \to B_x$ is a cofibration in $\mathcal{M}$ for all $x \in \mathcal{C}$
- A fibration iff the induced map $A_x \to B_x \times_{M_x B} M_x A$ is a fibration in $\mathcal{M}$ for all $x \in \mathcal{C}$.

We have the external product bifunctor $\mathsf{sSet} \times \mathsf{sSet} \xrightarrow{\square} \mathsf{ssSet}$, defined as $(A \square B)_{m,n} := A_m \times B_n$ which is biclosed. In particular, we have the adjoint pair

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "A\square\,\_"] \arrow[r, phantom, "\perp"] & \mathsf{ssSet} \arrow[l, bend left, "\{A,\,\_\}"] \end{tikzcd}$$

Weighted limit

Definition 3.1 (Weighted limit)

Given a simplicial object X in a locally small category $\mathcal{M}$ and a simplicial set S, define the weighted limit $\{S, X\}$ to be an object in $\mathcal{M}$ equipped with an isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(\_, \{S, X\}) \cong \mathrm{Hom}_{\mathsf{sSet}}(S, \mathrm{Hom}_{\mathcal{M}}(\_, X))$$

of functors $\mathcal{M}^{op} \to \mathsf{Set}$.

Weighted limit

Definition 3.1 (Weighted limit)

Given a simplicial object X in a locally small category $\mathcal{M}$ and a simplicial set S, define the weighted limit $\{S, X\}$ to be an object in $\mathcal{M}$ equipped with an isomorphism

$$\mathsf{Hom}_{\mathcal{M}}(\_, \{S, X\}) \cong \mathsf{Hom}_{\mathsf{sSet}}(S, \mathsf{Hom}_{\mathcal{M}}(\_, X))$$

of functors $\mathcal{M}^{op} \to \mathsf{Set}$.

In particular, if $\mathcal{M} = \mathsf{sSet}$, then

Weighted limit

Definition 3.1 (Weighted limit)

Given a simplicial object X in a locally small category $\mathcal{M}$ and a simplicial set S, define the weighted limit $\{S, X\}$ to be an object in $\mathcal{M}$ equipped with an isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(\_, \{S, X\}) \cong \mathrm{Hom}_{\mathsf{sSet}}(S, \mathrm{Hom}_{\mathcal{M}}(\_, X))$$

of functors $\mathcal{M}^{op} \to \mathsf{Set}$.

In particular, if $\mathcal{M} = \mathsf{sSet}$, then

$$\mathrm{Hom}_{\mathsf{sSet}}(S, \{\Delta^m, X\}) \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^m, \mathrm{Hom}_{\mathsf{sSet}}(S, X))$$
$$\mathrm{Hom}_{\mathsf{sSet}}(S, \{\Delta^m, X\}) \cong \mathrm{Hom}_{\mathsf{sSet}}(S, X(\_, [m]))$$
$$\{\Delta^m, X\} \cong X(\_, [m])$$

We denote $X(\_, [m])$ by X_m, the mth column of X.

Fibration vs Fibrant

Definition 3.2 (Reedy fibration)

A morphism $X \to Y$ is a Reedy fibration if and only if for all $m \geqslant 0$ the induced map

$$\{\Delta^m, X\} \to \{\partial\Delta^m, X\} \times_{\{\partial\Delta^m, Y\}} \{\Delta^m, Y\}$$

on weighted limits is a Kan fibration in sSet.

In the bisimplicial sets model, a dependent type family $C : A \to \mathcal{U}$ is modeled by a Reedy fibration $C \twoheadrightarrow A$.

Fibration vs Fibrant

Definition 3.2 (Reedy fibration)

A morphism $X \to Y$ is a Reedy fibration if and only if for all $m \geqslant 0$ the induced map

$$\{\Delta^m, X\} \to \{\partial\Delta^m, X\} \times_{\{\partial\Delta^m, Y\}} \{\Delta^m, Y\}$$

on weighted limits is a Kan fibration in sSet.

In the bisimplicial sets model, a dependent type family $C : A \to \mathcal{U}$ is modeled by a Reedy fibration $C \twoheadrightarrow A$.

Definition (Reedy fibrant)

A bisimplicial set X is Reedy fibrant just when the unique map $X \to 1$ is a Reedy fibration, which is the case when

$$\{\Delta^m, X\} \to \{\partial\Delta^m, X\}$$

is a Kan fibration.

In the bisimplicial sets model, a type is modeled by a Reedy fibrant bisimplicial set.

Pullback power axiom

Lemma 3.4 (Pullback power axiom)

If $i : U \to V$ is a cofibration and $p : X \twoheadrightarrow Y$ is a Reedy fibration then the map

$$\langle X^i, p^V \rangle : X^V \to X^U \times_{Y^U} Y^V,$$

which we denote by $\widehat{\{i, p\}}$, is a Reedy fibration, whose domain and codomain are Reedy fibrant if X and Y are, and which is a weak equivalence if p is.

Pullback power axiom

Lemma 3.4 (Pullback power axiom)

If $i : U \to V$ is a cofibration and $p : X \twoheadrightarrow Y$ is a Reedy fibration then the map
$$\langle X^i, p^V \rangle : X^V \to X^U \times_{Y^U} Y^V,$$
which we denote by $\widehat{\{i, p\}}$, is a Reedy fibration, whose domain and codomain are Reedy fibrant if X and Y are, and which is a weak equivalence if p is.

Proof. The key here is to use the equivalence of pullback power axiom and the pushout product axiom. We show that if $i : U \to V$ and $j : A \to B$ are cofibrations of bisimplicial sets, then $i \widehat{\times} j$ is cofibration that is trivial if j is.

Pullback power axiom

Lemma 3.4 (Pullback power axiom)

If $i : U \to V$ is a cofibration and $p : X \twoheadrightarrow Y$ is a Reedy fibration then the map

$$\langle X^i, p^V \rangle : X^V \to X^U \times_{Y^U} Y^V,$$

which we denote by $\widehat{\{i, p\}}$, is a Reedy fibration, whose domain and codomain are Reedy fibrant if X and Y are, and which is a weak equivalence if p is.

Proof. The key here is to use the equivalence of pullback power axiom and the pushout product axiom. We show that if $i : U \to V$ and $j : A \to B$ are cofibrations of bisimplicial sets, then $i \widehat{\times} j$ is cofibration that is trivial if j is.

$$\begin{tikzcd}
U \times A \arrow[r, "i\times 1_A"] \arrow[d, "1_U\times j"'] & V \times A \arrow[d, "k"'] \arrow[ddr, bend left, "1_V\times j"] & \\
U \times B \arrow[r] \arrow[drr, "i\times 1_B"'] & \star \arrow[dr, dashed, "i\widehat{\times}j"] & \\
 & & V \times B
\end{tikzcd}$$

Segal space

Definition 3.5 (Segal space)

A Reedy fibrant bisimplicial set X is a Segal space if and only if for all $m \geqslant 2$ and $0 < i < m$ the induced map,

$$\{\Delta^m, X\} \to \{\Lambda_i^m, X\}$$

on weighted limits is a trivial fibration in sSet.

Segal space

Definition 3.5 (Segal space)

A Reedy fibrant bisimplicial set X is a Segal space if and only if for all $m \geqslant 2$ and $0 < i < m$ the induced map,

$$\{\Delta^m, X\} \to \{\Lambda_i^m, X\}$$

on weighted limits is a trivial fibration in sSet.

Theorem 3.6 (Shulman)

The Reedy model structure on bisimplicial sets defined relative to the Quillen model structure on simplicial sets models intensional type theory with dependent sums, dependent products, identity types, and as many univalent universes as there are inaccessible cardinals greater than $\aleph_0$.

How to internalize the concept of arrows?

The fundamental structure we add to our HoTT is a directed interval type, which, thinking categorically, we denote $\mathbb{2}$.

How to internalize the concept of arrows?

The fundamental structure we add to our HoTT is a directed interval type, which, thinking categorically, we denote $\mathbb{2}$. As it does in ordinary category theory, the directed interval detects arrows representably, i.e., for any type A the function type $\mathbb{2} \to A$ is the type of arrows in A.

How to internalize the concept of arrows?

The fundamental structure we add to our HoTT is a directed interval type, which, thinking categorically, we denote $\mathbb{2}$. As it does in ordinary category theory, the directed interval detects arrows representably, i.e., for any type A the function type $\mathbb{2} \to A$ is the type of arrows in A.

- **But, how can one talk about the dependent type of arrows from x to y, for $x, y : A$?**

How to internalize the concept of arrows?

The fundamental structure we add to our HoTT is a directed interval type, which, thinking categorically, we denote $\mathbb{2}$. As it does in ordinary category theory, the directed interval detects arrows representably, i.e., for any type A the function type $\mathbb{2} \to A$ is the type of arrows in A.

- **But, how can one talk about the dependent type of arrows from x to y, for $x, y : A$?**

One natural way is to define this type internally by,

How to internalize the concept of arrows?

The fundamental structure we add to our HoTT is a directed interval type, which, thinking categorically, we denote $\mathbb{2}$. As it does in ordinary category theory, the directed interval detects arrows representably, i.e., for any type A the function type $\mathbb{2} \to A$ is the type of arrows in A.

• But, how can one talk about the dependent type of arrows from x to y, for $x, y : A$?

One natural way is to define this type internally by,

$$\hom_A(x, y) := \sum_{f:\mathbb{2}\to A} (x =_A f(0)) \times (f(1) =_A y)$$

How to internalize the concept of arrows?

The fundamental structure we add to our HoTT is a directed interval type, which, thinking categorically, we denote $\mathbb{2}$. As it does in ordinary category theory, the directed interval detects arrows representably, i.e., for any type A the function type $\mathbb{2} \to A$ is the type of arrows in A.

• **But, how can one talk about the dependent type of arrows from x to y, for $x, y : A$?**

One natural way is to define this type internally by,

$$\mathrm{hom}_A(x, y) := \sum_{f:\mathbb{2}\to A} (x =_A f(0)) \times (f(1) =_A y)$$

But these equalities are then data, which have to be carried around everywhere. This is quite tedious, and the technicalities become nearly insurmountable when we come to define commutative triangles and commuatative squares.

How to internalize the concept of arrows?

- **So, why not define $\hom_A(x, y)$ to be the type of functions $f : 2 \to A$ such that $x \equiv f(0)$ and $f(1) \equiv y$?**

How to internalize the concept of arrows?

- **So, why not define $\hom_A(x, y)$ to be the type of functions $f : 2 \to A$ such that $x \equiv f(0)$ and $f(1) \equiv y$?**

The first problem is that judgmental equality on A is interpreted by the diagonal $A \to A \times A$, which is usually not a fibration, unlike the path-object $PA \to A \times A$, which interprets the identity type, is a fibration.

How to internalize the concept of arrows?

• **So, why not define $\hom_A(x, y)$ to be the type of functions $f : 2 \to A$ such that $x \equiv f(0)$ and $f(1) \equiv y$?**

The first problem is that judgmental equality on A is interpreted by the diagonal $A \to A \times A$, which is usually not a fibration, unlike the path-object $PA \to A \times A$, which interprets the identity type, is a fibration.

However, since $\mathbf{2} \to 2$ is a cofibration and $A \to \mathbf{1}$ is a fibration, we obtain that the pullback corner map $A^2 \to A \times A$ is a fibration, which represents the the desired type family $\hom_A : A \times A \to \mathcal{U}$.

How to internalize the concept of arrows?

- **So, why not define hom$_A(x, y)$ to be the type of functions $f : 2 \to A$ such that $x \equiv f(0)$ and $f(1) \equiv y$?**

The first problem is that judgmental equality on A is interpreted by the diagonal $A \to A \times A$, which is usually not a fibration, unlike the path-object $PA \to A \times A$, which interprets the identity type, is a fibration.

However, since $\mathbf{2} \to 2$ is a cofibration and $A \to \mathbf{1}$ is a fibration, we obtain that the pullback corner map $A^2 \to A \times A$ is a fibration, which represents the the desired type family $\mathrm{hom}_A : A \times A \to \mathcal{U}$.

We instead use a more refined approach where we have a judgemental notion of cofibration, and a new type former called an extension type: if $i : A \rightarrowtail B$ is a cofibration and $C : B \to \mathcal{U}$ is a type family with a section $d : \prod_{x:A} C(i(x))$, then there is a type $\left\langle \prod_{y:B} C(y) \Big|_d^i \right\rangle$ of dependent functions $f : \prod_{y:B} C(y)$ such that $f(i(x)) \equiv d(x)$ for all $x : A$.

The cube layer

Concretely, sHoTT is built as a three layer type theory: The first layer is the coherent theory of cubes with finite products of cubes and an axiomatic cube $\mathbb{2}$ with no other data.

The cube layer

Concretely, sHoTT is built as a three layer type theory: The first layer is the coherent theory of cubes with finite products of cubes and an axiomatic cube $\mathbb{2}$ with no other data.

So here is our first layer: the layer of **cubes**

$$\frac{}{\mathbf{1}\ \text{cube}} \qquad \frac{}{\mathbb{2}\ \text{cube}} \qquad \frac{I\ \text{cube} \quad J\ \text{cube}}{I \times J\ \text{cube}} \qquad \frac{(t : I) \in \Xi}{\Xi \vdash t : I} \qquad \frac{}{\Xi \vdash \star : I}$$

$$\frac{\Xi \vdash s : I \quad \Xi \vdash t : J}{\Xi \vdash \langle s, t \rangle : I \times J} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \vdash \pi_1(t) : I} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \vdash \pi_2(t) : J}$$

Here Ξ is a context of variables belonging to cubes, and $\mathbf{1}$ denotes the empty product.

The tope layer

The second layer is an intuitionistic logic over the layer of cubes. We refer to its types as **topes** admitting finite conjunction and disjunction, but not negation, implication, or either quantifier.

The tope layer

The second layer is an intuitionistic logic over the layer of cubes. We refer to its types as **topes** admitting finite conjunction and disjunction, but not negation, implication, or either quantifier.

Here is our second layer: the layer of topes

$$\frac{\phi \in \Phi}{\Xi \mid \Phi \vdash \phi} \qquad \frac{}{\Xi \vdash \top \ \mathsf{tope}} \qquad \frac{}{\Xi \mid \Phi \vdash \top} \qquad \frac{}{\Xi \vdash \bot \ \mathsf{tope}} \qquad \frac{\Xi \mid \Phi \vdash \bot}{\Xi \mid \Phi \vdash \psi}$$

$$\frac{\Xi \vdash \phi \ \mathsf{tope} \quad \Xi \vdash \psi \ \mathsf{tope}}{\Xi \vdash (\phi \wedge \psi) \ \mathsf{tope}} \qquad \frac{\Xi \mid \Phi \vdash \phi \quad \Xi \mid \Phi \vdash \psi}{\Xi \mid \Phi \vdash \phi \wedge \psi} \qquad \frac{\Xi \mid \Phi \vdash \phi \wedge \psi}{\Xi \mid \Phi \vdash \phi}$$

$$\frac{\Xi \mid \Phi \vdash \phi \wedge \psi}{\Xi \mid \Phi \vdash \psi} \qquad \frac{\Xi \vdash \phi \ \mathsf{tope} \quad \Xi \vdash \psi \ \mathsf{tope}}{\Xi \vdash (\phi \vee \psi) \ \mathsf{tope}} \qquad \frac{\Xi \mid \Phi \vdash \phi}{\Xi \mid \Phi \vdash \phi \vee \psi}$$

The tope layer

$$\frac{\Xi \mid \Phi \vdash \psi}{\Xi \mid \Phi \vdash \phi \vee \psi} \qquad \frac{\Xi \mid \Phi, \phi \vdash \chi \quad \Xi \mid \phi, \psi \vdash \chi \quad \Xi \mid \Phi \vdash \phi \vee \psi}{\Xi \mid \Phi \vdash \chi}$$

$$\frac{\Xi \vdash s : I \quad \Xi \vdash t : I}{\Xi \vdash (s \equiv t) \text{ tope}} \qquad \frac{\Xi \vdash s : I}{\Xi \mid \Phi \vdash (s \equiv s)} \qquad \frac{\Xi \mid \Phi \vdash (s \equiv t)}{\Xi \mid \Phi \vdash (t \equiv s)}$$

$$\frac{\Xi \mid \Phi \vdash (s \equiv t) \quad \Xi \mid \Phi \vdash (t \equiv v)}{\Xi \mid \Phi \vdash (s \equiv v)} \qquad \frac{\Xi \vdash t : I}{\Xi \mid \Phi \vdash t \equiv \star}$$

$$\frac{\Xi \mid \Phi \vdash (s \equiv t) \quad \Xi, x : I \vdash \psi \text{ tope} \quad \Xi \mid \Phi \vdash \psi[s/x]}{\Xi \mid \Phi \vdash \psi[t/x]} \qquad \frac{\Xi \vdash s : I \quad \Xi \vdash t : J}{\Xi \mid \Phi \vdash \pi_1(\langle s, t \rangle) \equiv s}$$

$$\frac{\Xi \vdash s : I \quad \Xi \vdash t : J}{\Xi \mid \Phi \vdash \pi_2(\langle s, t \rangle) \equiv t} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \mid \Phi \vdash t \equiv \langle \pi_1(t), \pi_2(t) \rangle}$$

Here Φ is a list of topes, and $\equiv$ is an equality tope.

Shapes as special types

A shape will mean a cube together with a tope in the corresponding singleton context:

Shapes as special types

A shape will mean a cube together with a tope in the corresponding singleton context:

$$\frac{I \text{ cube} \quad t : I \vdash \phi \text{ tope}}{\{t : I \mid \phi\} \text{ shape}}$$

Shapes as special types

A shape will mean a cube together with a tope in the corresponding singleton context:

$$\frac{I \text{ cube} \quad t : I \vdash \phi \text{ tope}}{\{t : I \mid \phi\} \text{ shape}}$$

Some most relevant shapes for us will be the n-simplices, their boundaries and their horns.

$$\Delta^0 := \{t : 1 \mid \top\}$$
$$\Delta^1 := \{t : 2 \mid \top\}$$
$$\Delta^2 := \{\langle t_1, t_2 \rangle : 2 \times 2 \mid t_2 \leqslant t_1\}$$
$$\Delta^3 := \{\langle t_1, t_2, t_3 \rangle : 2 \times 2 \times 2 \mid t_3 \leqslant t_2 \leqslant t_1\}$$
$$\partial\Delta^1 := \{t : 2 \mid (t \equiv 0) \vee (t \equiv 1)\}$$
$$\partial\Delta^2 := \{(t_1, t_2) : \Delta^2 \mid (0 \equiv t_2 \leqslant t_1) \vee (t_1 \equiv t_2) \vee (t_2 \leqslant t_1 \equiv 1)\}$$
$$\Lambda^2_1 := \{\langle t_1, t_2 \rangle : 2 \times 2 \mid t_1 = 1 \vee t_2 = 0\}$$

Extension type

Finally, there is a third layer of types that has all the ordinary type formers of HoTT and one additional type former, the extension type.

Extension type

Finally, there is a third layer of types that has all the ordinary type formers of HoTT and one additional type former, the extension type.

$$\frac{\{t : I \mid \phi\}\ \text{shape} \quad \{t : I \mid \psi\}\ \text{shape} \quad t : I \mid \phi \vdash \psi \qquad \Xi \mid \Phi \vdash \Gamma\ \text{ctx} \quad \Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash A\ \text{type} \quad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A}{\Xi \mid \Phi \mid \Gamma \vdash \left\langle \prod_{t:I\mid\psi} A\Big|_{a}^{\phi} \right\rangle\ \text{type}}$$

Extension type

Finally, there is a third layer of types that has all the ordinary type formers of HoTT and one additional type former, the extension type.

$$\frac{\{t : I \mid \phi\} \text{ shape} \quad \{t : I \mid \psi\} \text{ shape} \quad t : I \mid \phi \vdash \psi \atop \Xi \mid \Phi \vdash \Gamma \text{ ctx} \qquad \Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash A \text{ type} \qquad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A}{\Xi \mid \Phi \mid \Gamma \vdash \left\langle \prod_{t:I\mid\psi} A\Big|_a^\phi \right\rangle \text{ type}}$$

$$\frac{\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash b : A \qquad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash b \equiv a}{\Xi \mid \Phi \mid \Gamma \vdash \lambda t^{I\mid\psi}.b : \left\langle \prod_{t:I\mid\psi} A\Big|_a^\phi \right\rangle}$$

Extension type

Finally, there is a third layer of types that has all the ordinary type formers of HoTT and one additional type former, the extension type.

$$\frac{\{t : I \mid \phi\}\ \text{shape} \quad \{t : I \mid \psi\}\ \text{shape} \quad t : I \mid \phi \vdash \psi}{\dfrac{\Xi \mid \Phi \vdash \Gamma\ \text{ctx} \qquad \Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash A\ \text{type} \qquad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A}{\Xi \mid \Phi \mid \Gamma \vdash \left\langle \prod_{t:I\mid\psi} A\big|^{\phi}_{a} \right\rangle\ \text{type}}}$$

$$\frac{\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash b : A \qquad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash b \equiv a}{\Xi \mid \Phi \mid \Gamma \vdash \lambda t^{I\mid\psi}.b : \left\langle \prod_{t:I\mid\psi} A\big|^{\phi}_{a} \right\rangle}$$

$$\frac{\Xi \mid \Phi \mid \Gamma \vdash f : \left\langle \prod_{t:I\mid\psi} A\big|^{\phi}_{a} \right\rangle \qquad \Xi \vdash s : I \quad \Xi \mid \Phi \vdash \psi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash f(s) : A}$$

Extension type

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \phi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash f(s) \equiv a[s/t]}$$

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \psi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash (\lambda t^{I|\psi}.b)(s) \equiv b[s/t]}$$

Extension type

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \phi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash f(s) \equiv a[s/t]}$$

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \psi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash (\lambda t^{I|\psi}.b)(s) \equiv b[s/t]}$$

We can think of $\{t : I \mid \phi\}$ as a sub-shape of $\{t : I \mid \psi\}$ and read the judgment $\Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A$ as a function $\phi \to A$, we could represent a point in an extension type with a dashed arrow in the commutative diagram:

Extension type

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \phi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash f(s) \equiv a[s/t]}$$

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \psi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash (\lambda t^{I|\psi}.b)(s) \equiv b[s/t]}$$

We can think of $\{t : I \mid \phi\}$ as a sub-shape of $\{t : I \mid \psi\}$ and read the judgment $\Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A$ as a function $\phi \to A$, we could represent a point in an extension type with a dashed arrow in the commutative diagram:

$$\begin{tikzcd} \phi \arrow[r, "a"] \arrow[d, hook] & A \\ \psi \arrow[ur, dashed] & \end{tikzcd}$$

Equivalences involving extension types

Just like for ordinary and dependent function types, where we have,

$$\left(\prod_{x:X}\prod_{y:Y} Z(x,y)\right) \simeq \left(\prod_{y:Y}\prod_{x:X} Z(x,y)\right).$$

Equivalences involving extension types

Just like for ordinary and dependent function types, where we have,

$$\left(\prod_{x:X}\prod_{y:Y} Z(x,y)\right) \simeq \left(\prod_{y:Y}\prod_{x:X} Z(x,y)\right).$$

The following theorem is an analogue for extension types.

Equivalences involving extension types

Just like for ordinary and dependent function types, where we have,

$$\left(\prod_{x:X}\prod_{y:Y} Z(x,y)\right) \simeq \left(\prod_{y:Y}\prod_{x:X} Z(x,y)\right).$$

The following theorem is an analogue for extension types.

Theorem 5.1 (Commutation and currying)

If $t : I \mid \phi \vdash \psi$ and $X : \mathcal{U}$, while $Y : \{t : I \mid \psi\} \to X \to \mathcal{U}$ and $f : \prod_{t:I|\phi}\prod_{x:X} Y(t,x)$, then

$$\left\langle \prod_{t:I|\psi} \left(\prod_{x:X} Y(t,x)\right)\Big|^{\phi}_{f} \right\rangle \simeq \prod_{x:X} \left\langle \prod_{t:I|\psi} Y(t,x)\Big|^{\phi}_{\lambda t.f(t,x)} \right\rangle.$$

Equivalences involving extension types

Just like for ordinary and dependent function types, where we have,

$$\left(\prod_{x:X}\prod_{y:Y} Z(x,y)\right) \simeq \left(\prod_{y:Y}\prod_{x:X} Z(x,y)\right).$$

The following theorem is an analogue for extension types.

Theorem 5.1 (Commutation and currying)

If $t : I \mid \phi \vdash \psi$ and $X : \mathcal{U}$, while $Y : \{t : I \mid \psi\} \to X \to \mathcal{U}$ and $f : \prod_{t:I|\phi}\prod_{x:X} Y(t,x)$, then

$$\left\langle \prod_{t:I|\psi}\left(\prod_{x:X} Y(t,x)\right)\Big|^{\phi}_{f}\right\rangle \simeq \prod_{x:X}\left\langle \prod_{t:I|\psi} Y(t,x)\Big|^{\phi}_{\lambda t.f(t,x)}\right\rangle.$$

Proof. From left to right $g \mapsto \lambda x.\lambda t.g(t,x)$, and from right to left $h \mapsto \lambda t.\lambda x.h(x,t)$. If $g(t) \equiv f(t)$ assuming ϕ, then $g(t,x) \equiv f(t,x)$ assuming ϕ. For the reverse direction, using η-conversion, if $h(x,t) \equiv f(t,x)$ assuming ϕ, then $\lambda t.\lambda x.h(x,t) \equiv \lambda t.\lambda x.f(t,x) \equiv f$ assuming ϕ.

Equivalences involving extension types

Theorem 5.2 (Pushout product)

If $t : I \mid \phi \vdash \psi$ and $s : J \mid \chi \vdash \zeta$, while $X : \{t : I \mid \psi\} \to \{s : J \mid \zeta\} \to \mathcal{U}$ and $f : \prod_{<t,s>:I\times J|(\phi\wedge\zeta)\vee(\psi\wedge\chi)} X(t,s)$, then

$$\left\langle \prod_{t:I|\psi} \left\langle \prod_{s:J|\zeta} X(t,s) \Big|^{\chi}_{\lambda s.f(t,s)} \right\rangle \Big|^{\phi}_{\lambda t.\lambda s.f<t,s>} \right\rangle$$

$$\simeq \left\langle \prod_{<t,s>:I\times J|\psi\wedge\zeta} X(t,s) \Big|^{(\phi\wedge\zeta)\vee(\psi\wedge\chi)}_{f} \right\rangle$$

$$\simeq \left\langle \prod_{s:J|\zeta} \left\langle \prod_{t:I|\psi} X(t,s) \Big|^{\phi}_{\lambda s.f(t,s)} \right\rangle \Big|^{\chi}_{\lambda t.\lambda s.f<t,s>} \right\rangle.$$

Equivalences involving extension types

Theorem 5.2 (Pushout product)

If $t : I \mid \phi \vdash \psi$ and $s : J \mid \chi \vdash \zeta$, while $X : \{t : I \mid \psi\} \to \{s : J \mid \zeta\} \to \mathcal{U}$ and $f : \prod_{<t,s>:I\times J|(\phi\wedge\zeta)\vee(\psi\wedge\chi)} X(t,s)$, then

$$\left\langle \prod_{t:I|\psi} \left\langle \prod_{s:J|\zeta} X(t,s) \Big|^{\chi}_{\lambda s.f(t,s)} \right\rangle \Big|^{\phi}_{\lambda t.\lambda s.f<t,s>} \right\rangle$$

$$\simeq \left\langle \prod_{<t,s>:I\times J|\psi\wedge\zeta} X(t,s) \Big|^{(\phi\wedge\zeta)\vee(\psi\wedge\chi)}_{f} \right\rangle$$

$$\simeq \left\langle \prod_{s:J|\zeta} \left\langle \prod_{t:I|\psi} X(t,s) \Big|^{\phi}_{\lambda s.f(t,s)} \right\rangle \Big|^{\chi}_{\lambda t.\lambda s.f<t,s>} \right\rangle.$$

The shape $\{\langle t,s\rangle : I \times J \mid (\phi\wedge\zeta)\vee(\psi\wedge\chi)\}$ may be called the pushout product of the two inclusions $\{t : I \mid \phi\} \subseteq \{t : I \mid \psi\}$ and $\{s : J \mid \chi\} \subseteq \{s : J \mid \zeta\}$. The dashed arrow in the diagram may be called the pushout product map.

Equivalences involving extension types

Consider the pushout square,

$$\begin{tikzcd}
\psi \wedge \zeta & & \\
& (\phi \wedge \zeta) \vee (\psi \wedge \chi) \arrow[ul, dashed] & \phi \wedge \zeta \arrow[l] \arrow[ull, bend right] \\
& \psi \wedge \chi \arrow[u] \arrow[uul, bend left] & \phi \wedge \chi \arrow[l] \arrow[u]
\end{tikzcd}$$

Equivalences involving extension types

Consider the pushout square,

$$\begin{tikzcd}
\psi \wedge \zeta & & \\
& (\phi \wedge \zeta) \vee (\psi \wedge \chi) \arrow[ul, dashed] & \phi \wedge \zeta \arrow[l] \arrow[ull, bend right] \\
& \psi \wedge \chi \arrow[u] \arrow[uul, bend left] & \phi \wedge \chi \arrow[l] \arrow[u]
\end{tikzcd}$$

The first and the third is just application and re-abstraction. For the first with the second, we perform currying, i.e., from left to right $\lambda t.\lambda s.f\langle t, s\rangle \mapsto \lambda\langle t, s\rangle.f\langle t, s\rangle$ and right to left $\lambda\langle t, s\rangle.f\langle t, s\rangle \mapsto \lambda t.\lambda s.f\langle t, s\rangle$. Similarly, the second and the third are equivalent.

Equivalences involving extension types

Theorem 5.3 (Principle of choice)

If $t : I \mid \phi \vdash \psi$, while $X : \{t : I \mid \psi\} \to \mathcal{U}$ and $Y : \prod_{t:I|\psi}(X \to \mathcal{U})$, while $a : \prod_{t:I|\phi} X(t)$ and $b : \prod t : I \mid \phi Y(t, x(t))$, then

$$\left\langle \prod_{t:I|\psi} \left(\sum_{x:X(t)} Y(t,x) \right) \Big|^{\phi}_{\lambda t.(a(t),b(t))} \right\rangle \simeq$$

$$\sum_{f:\left\langle \prod_{t:I|\psi} X(t) \big|^{\phi}_{a} \right\rangle} \left\langle \prod_{t:I|\psi} Y(t, f(t)) \big|^{\phi}_{b} \right\rangle.$$

Equivalences involving extension types

Theorem 5.3 (Principle of choice)

If $t : I \mid \phi \vdash \psi$, while $X : \{t : I \mid \psi\} \to \mathcal{U}$ and $Y : \prod_{t:I|\psi}(X \to \mathcal{U})$, while $a : \prod_{t:I|\phi} X(t)$ and $b : \prod t : I \mid \phi Y(t, x(t))$, then

$$\left\langle \prod_{t:I|\psi} \left(\sum_{x:X(t)} Y(t,x) \right) \Big|^{\phi}_{\lambda t.(a(t),b(t))} \right\rangle \simeq \sum_{f:\left\langle \prod_{t:I|\psi} X(t) \big|^{\phi}_{a} \right\rangle} \left\langle \prod_{t:I|\psi} Y(t, f(t)) \big|^{\phi}_{b} \right\rangle.$$

Proof. As in the ordinary case, this is just composing the introduction and elimination rules. Again, from left to right, $h \mapsto (\lambda t.\pi_1(h(t)), \lambda t.\pi_2(h(t)))$ and from right to left, $(f, g) \mapsto \lambda t.(f(t), g(t))$. The β-reduction and η-expansion rules make these inverse equivalences.

Equivalences involving extension types

Theorem 5.4 (Composition)

Suppose $t : T \mid \phi \vdash \psi$ and $t : I \mid \psi \vdash \chi$, and that $X : \{t : I \mid \chi\} \to \mathcal{U}$ and $a : \prod_{t:I|\phi} X(t)$. Then, $\left\langle \prod_{t:I|\chi} X\Big|_a^\phi \right\rangle \simeq \left(\sum_{f:\left\langle \prod_{t:I|\chi} X\big|_a^\phi \right\rangle} \left\langle \prod_{t:I|\chi} X\Big|_f^\psi \right\rangle \right)$.

Equivalences involving extension types

Theorem 5.4 (Composition)

Suppose $t : T \mid \phi \vdash \psi$ and $t : I \mid \psi \vdash \chi$, and that $X : \{t : I \mid \chi\} \to \mathcal{U}$ and $a : \prod_{t:I|\phi} X(t)$. Then, $\left\langle \prod_{t:I|\chi} X\Big|^{\phi}_{a} \right\rangle \simeq \left(\sum_{f:\left\langle \prod_{t:I|\chi} X\big|^{\phi}_{a} \right\rangle} \left\langle \prod_{t:I|\chi} X\Big|^{\psi}_{f} \right\rangle \right)$.

Proof. From left to right, $h \mapsto (\lambda t.h(t), \lambda t.h(t))$ and from right to left, $(f, g) \mapsto \lambda t.g(t)$.

Equivalences involving extension types

Theorem 5.4 (Composition)

Suppose $t : T \mid \phi \vdash \psi$ and $t : I \mid \psi \vdash \chi$, and that $X : \{t : I \mid \chi\} \to \mathcal{U}$ and $a : \prod_{t:I|\phi} X(t)$. Then, $\left\langle \prod_{t:I|\chi} X\Big|_a^\phi \right\rangle \simeq \left(\sum_{f:\left\langle \prod_{t:I|\chi} X\big|_a^\phi \right\rangle} \left\langle \prod_{t:I|\chi} X\Big|_f^\psi \right\rangle \right)$.

Proof. From left to right, $h \mapsto (\lambda t.h(t), \lambda t.h(t))$ and from right to left, $(f, g) \mapsto \lambda t.g(t)$.

Theorem 5.5 (Union)

Suppose $t : I \vdash \phi$ tope and $t : I \vdash \psi$ tope, and that we have $X : \{t : I \mid \phi \vee \psi\} \to \mathcal{U}$ and $a : \prod_{t:I|\psi} X(t)$. Then,

$$\left\langle \prod_{t:I|\phi\vee\psi} X\Big|_a^\psi \right\rangle \simeq \left\langle \prod_{t:I|\phi} X\Big|_{\lambda t.a(t)}^{\phi\wedge\psi} \right\rangle.$$

Equivalences involving extension types

Theorem 5.4 (Composition)

Suppose $t : T \mid \phi \vdash \psi$ and $t : I \mid \psi \vdash \chi$, and that $X : \{t : I \mid \chi\} \to \mathcal{U}$ and $a : \prod_{t:I|\phi} X(t)$. Then, $\left\langle \prod_{t:I|\chi} X\Big|_a^\phi \right\rangle \simeq \left(\sum_{f:\left\langle \prod_{t:I|\chi} X\big|_a^\phi \right\rangle} \left\langle \prod_{t:I|\chi} X\Big|_f^\psi \right\rangle \right)$.

Proof. From left to right, $h \mapsto (\lambda t.h(t), \lambda t.h(t))$ and from right to left, $(f, g) \mapsto \lambda t.g(t)$.

Theorem 5.5 (Union)

Suppose $t : I \vdash \phi$ tope and $t : I \vdash \psi$ tope, and that we have $X : \{t : I \mid \phi \vee \psi\} \to \mathcal{U}$ and $a : \prod_{t:I|\psi} X(t)$. Then,

$$\left\langle \prod_{t:I|\phi\vee\psi} X\Big|_a^\psi \right\rangle \simeq \left\langle \prod_{t:I|\phi} X\Big|_{\lambda t.a(t)}^{\phi\wedge\psi} \right\rangle.$$

Proof. Again, frpm left to right, we re-package, $h \mapsto \lambda t.h(t)$. From right to left, $g \mapsto \lambda t.rec_\vee^{\phi,\psi}(g(t), a(t))$, is well defined since $g(t) \equiv a(t)$ for $t : I$ satisfying $\phi \wedge \psi$.

Equivalences involving extension types

Now, we introduce the function extensionality for extension types. Before doing that, let us recall the three equivalent formulations of function extensionality for ordinary dependent functions:

Equivalences involving extension types

Now, we introduce the function extensionality for extension types. Before doing that, let us recall the three equivalent formulations of function extensionality for ordinary dependent functions:

- For $f, g : \prod_{x:A} B(x)$ if $\prod_{x:A}(fx = gx)$, then $f = g$.

Equivalences involving extension types

Now, we introduce the function extensionality for extension types. Before doing that, let us recall the three equivalent formulations of function extensionality for ordinary dependent functions:

- For $f, g : \prod_{x:A} B(x)$ if $\prod_{x:A}(fx = gx)$, then $f = g$.
- For $f, g : \prod_{x:A} B(x)$, the canonical map $(f = g) \to \prod_{x:A}(fx = gx)$ is an equivalence.

Equivalences involving extension types

Now, we introduce the function extensionality for extension types. Before doing that, let us recall the three equivalent formulations of function extensionality for ordinary dependent functions:

- For $f, g : \prod_{x:A} B(x)$ if $\prod_{x:A}(fx = gx)$, then $f = g$.
- For $f, g : \prod_{x:A} B(x)$, the canonical map $(f = g) \to \prod_{x:A}(fx = gx)$ is an equivalence.
- If each $B(x)$ is contractible, then so is $\prod_{x:A} B(x)$.

Equivalences involving extension types

Now, we introduce the function extensionality for extension types. Before doing that, let us recall the three equivalent formulations of function extensionality for ordinary dependent functions:

- For $f, g : \prod_{x:A} B(x)$ if $\prod_{x:A}(fx = gx)$, then $f = g$.
- For $f, g : \prod_{x:A} B(x)$, the canonical map $(f = g) \to \prod_{x:A}(fx = gx)$ is an equivalence.
- If each $B(x)$ is contractible, then so is $\prod_{x:A} B(x)$.

Axiom 6.5 (Relative functional extensionality)

Supposing $t : I \mid \phi \vdash \psi$ and that $A : \{t : I \mid \psi\} \to \mathcal{U}$ is such that each $A(t)$ is contractible, and moreover $a : \prod_{t:I|\phi} A(t)$, then $\left\langle \prod_{t:I|\psi} a(t) \Big|_{a}^{\phi} \right\rangle$ is contractible.

Equivalences involving extension types

Proposition 5.6 (Homotopy extension property)

Let $t : I \mid \phi \vdash \psi$. Assuming the relative function extensionality, if we have $A : \{t : I \mid \psi\} \to \mathcal{U}$ and $b : \prod_{t:I|\psi} A(t)$, and moreover $a : \prod_{t:I|\phi} A(t)$ and $e : \prod_{t:I|\phi} a(t) = b(t)$, then we have $a' : \left\langle \prod_{t:I|\psi} A(t)\Big|^{\phi}_{a} \right\rangle$ and $e' : \left\langle \prod_{t:I|\psi} a'(t) = b(t)\Big|^{\phi}_{e} \right\rangle$.

Equivalences involving extension types

Proposition 5.6 (Homotopy extension property)

Let $t : I \mid \phi \vdash \psi$. Assuming the relative function extensionality, if we have $A : \{t : I \mid \psi\} \to \mathcal{U}$ and $b : \prod_{t:I|\psi} A(t)$, and moreover $a : \prod_{t:I|\phi} A(t)$ and $e : \prod_{t:I|\phi} a(t) = b(t)$, then we have $a' : \left\langle \prod_{t:I|\psi} A(t)\Big|^{\phi}_{a} \right\rangle$ and $e' : \left\langle \prod_{t:I|\psi} a'(t) = b(t)\Big|^{\phi}_{e} \right\rangle$.

Proof. The extension type $\left\langle \prod_{t:I|\psi} \left(\sum_{y:A(t)} (y = b(t)) \right)\Big|^{\phi}_{\lambda t.(a(t),e(t))} \right\rangle$ is contractible by the axiom of relative function extensionality, hence inhabited.

Equivalences involving extension types

Proposition 5.6 (Homotopy extension property)

Let $t : I \mid \phi \vdash \psi$. Assuming the relative function extensionality, if we have $A : \{t : I \mid \psi\} \to \mathcal{U}$ and $b : \prod_{t:I|\psi} A(t)$, and moreover $a : \prod_{t:I|\phi} A(t)$ and $e : \prod_{t:I|\phi} a(t) = b(t)$, then we have $a' : \left\langle \prod_{t:I|\psi} A(t)\Big|^{\phi}_{a} \right\rangle$ and $e' : \left\langle \prod_{t:I|\psi} a'(t) = b(t)\Big|^{\phi}_{e} \right\rangle$.

Proof. The extension type $\left\langle \prod_{t:I|\psi} \left(\sum_{y:A(t)} (y = b(t)) \right)\Big|^{\phi}_{\lambda t.(a(t),e(t))} \right\rangle$ is contractible by the axiom of relative function extensionality, hence inhabited. Finally, we obtain a' and e' by applying Theorem 5.3.

Arrows as terms of extension type

Definition 6.1 (Directed hom-type)

Given $x, y : A$, determining a term $[x, y] : A$ in context $\partial\Delta^1$, we define,

$$\mathsf{hom}_A(x, y) := \left\langle \Delta^1 \to A \Big|^{\partial\Delta^1}_{[x,y]} \right\rangle$$

Arrows as terms of extension type

Definition 6.1 (Directed hom-type)

Given $x, y : A$, determining a term $[x, y] : A$ in context $\partial\Delta^1$, we define,

$$\hom_A(x, y) := \left\langle \Delta^1 \to A \Big|^{\partial\Delta^1}_{[x,y]} \right\rangle$$

We refer to an element of $\hom_A(x, y)$ as an arrow from x to y in A. Every $f : \hom_A(x, y)$ is a kind of function from 2 to A, with the property that $f(0) \equiv x$ and $f(1) \equiv y$.

Arrows as terms of extension type

Definition 6.1 (Directed hom-type)

Given $x, y : A$, determining a term $[x, y] : A$ in context $\partial\Delta^1$, we define,

$$\mathsf{hom}_A(x, y) := \left\langle \Delta^1 \to A \Big|^{\partial\Delta^1}_{[x,y]} \right\rangle$$

We refer to an element of $\mathsf{hom}_A(x, y)$ as an arrow from x to y in A. Every $f : \hom_A(x, y)$ is a kind of function from 2 to A, with the property that $f(0) \equiv x$ and $f(1) \equiv y$.

Definition 6.2 (Composition type)

Given $x, y, z : A$ and $f : \hom_A(x, y)$, $g : \hom_A(y, z)$ and $h : \hom_A(x, z)$ we have an induced term $[x, y, z, f, g, h] : A$ in context $\partial\Delta^2$, and an extension type that we denote,

$$\hom^2_A \left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \right) := \left\langle \Delta^2 \to A \Big|^{\partial\Delta^2}_{[x,y,z,f,g,h]} \right\rangle$$

Segal types are special

Definition 6.3 (Segal type)

A **Segal type** is a type A such that for all $x, y, z : A$ and $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$ the type,

$$\sum_{h:\hom_A(x,z)} \hom^2_A\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}\right)$$

is contractible.

Segal types are special

Definition 6.3 (Segal type)

A **Segal type** is a type A such that for all $x, y, z : A$ and $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$ the type,

$$\sum_{h : \hom_A(x,z)} \hom^2_A \left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \right)$$

is contractible.

In particular, the above type is inhabited, and the first component of this inhabitant is denoted by $g \circ f : \hom_A(x, z)$, the composite of g and f. The second component of this inhabitant is a 2-simplex in $\hom^2_A(f, g, g \circ f)$, denoted by $\mathsf{comp}_{g,f}$.

Segal types are special

Definition 6.3 (Segal type)

A **Segal type** is a type A such that for all $x, y, z : A$ and $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$ the type,

$$\sum_{h:\hom_A(x,z)} \hom_A^2 \left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \right)$$

is contractible.

In particular, the above type is inhabited, and the first component of this inhabitant is denoted by $g \circ f : \hom_A(x, z)$, the composite of g and f. The second component of this inhabitant is a 2-simplex in $\hom_A^2(f, g, g \circ f)$, denoted by $\mathsf{comp}_{g,f}$. One can reformulate the above type as a single extension type of functions $\Delta^2 \to A$ that restrict to f and g on

$$\Lambda_1^2 = \{\langle s, t\rangle : 2 \times 2 \mid s \equiv 1 \vee t \equiv 0\}$$

Does Segal types $\leftrightarrow$ Segal spaces really?

Theorem 6.4 (Filling of 2-dimensional horns)

A type A is **Segal** if and only if the restriction map,

$$(\Delta^2 \to A) \to (\Lambda_1^2 \to A)$$

is an equivalence.

Does Segal types $\leftrightarrow$ Segal spaces really?

Theorem 6.4 (Filling of 2-dimensional horns)

A type A is **Segal** if and only if the restriction map,

$$(\Delta^2 \to A) \to (\Lambda_1^2 \to A)$$

is an equivalence.

Proof. By theorem 5.5, to extend a map $\Lambda_1^2 \to A$ to $\partial\Delta^2$ is equivalent to extending its restriction to $\partial\Delta_1^1$ to $\Delta_1^1 := \{\langle s,t\rangle : 2 \times 2 \mid s = t\}$.

Does Segal types $\leftrightarrow$ Segal spaces really?

Theorem 6.4 (Filling of 2-dimensional horns)

A type A is **Segal** if and only if the restriction map,
$$(\Delta^2 \to A) \to (\Lambda_1^2 \to A)$$
is an equivalence.

Proof. By theorem 5.5, to extend a map $\Lambda_1^2 \to A$ to $\partial\Delta^2$ is equivalent to extending its restriction to $\partial\Delta_1^1$ to $\Delta_1^1 := \{\langle s,t\rangle : 2 \times 2 \mid s = t\}$.

$$\sum_{h:\left\langle \Delta^1 \to A\Big|^{\partial\Delta^1}_{[x,z]} \right\rangle} \left\langle \Delta^2 \to A\Big|^{\partial\Delta^2}_{[x,y,z,f,g,h]} \right\rangle \simeq \sum_{l:\left\langle \partial\Delta^2 \to A\Big|^{\Lambda_1^2}_{[x,y,z,f,g]} \right\rangle} \left\langle \Delta^2 \to A\Big|^{\partial\Delta^2}_{l} \right\rangle \simeq \left\langle \Delta^2 \to A\Big|^{\Lambda_1^2}_{[x,y,z,f,g]} \right\rangle.$$

Does Segal types $\leftrightarrow$ Segal spaces really?

Theorem 6.4 (Filling of 2-dimensional horns)

A type A is **Segal** if and only if the restriction map,

$$(\Delta^2 \to A) \to (\Lambda_1^2 \to A)$$

is an equivalence.

Proof. By theorem 5.5, to extend a map $\Lambda_1^2 \to A$ to $\partial\Delta^2$ is equivalent to extending its restriction to $\partial\Delta_1^1$ to $\Delta_1^1 := \{\langle s,t\rangle : 2 \times 2 \mid s = t\}$.

$$\sum_{h:\left\langle \Delta^1 \to A\middle|^{\partial\Delta^1}_{[x,z]}\right\rangle} \left\langle \Delta^2 \to A\middle|^{\partial\Delta^2}_{[x,y,z,f,g,h]}\right\rangle \simeq \sum_{l:\left\langle \partial\Delta^2 \to A\middle|^{\Lambda_1^2}_{[x,y,z,f,g]}\right\rangle} \left\langle \Delta^2 \to A\middle|^{\partial\Delta^2}_{l}\right\rangle \simeq$$

$\left\langle \Delta^2 \to A\middle|^{\Lambda_1^2}_{[x,y,z,f,g]}\right\rangle$. Using 5.4, $(\Delta^2 \to A) \simeq \sum_{k:\Lambda_1^2 \to A} \left\langle \Delta^2 \to A\middle|^{\Lambda_1^2}_{k}\right\rangle$.

Does Segal types $\leftrightarrow$ Segal spaces really?

Theorem 6.4 (Filling of 2-dimensional horns)

A type A is **Segal** if and only if the restriction map,
$$(\Delta^2 \to A) \to (\Lambda_1^2 \to A)$$
is an equivalence.

Proof. By theorem 5.5, to extend a map $\Lambda_1^2 \to A$ to $\partial\Delta^2$ is equivalent to extending its restriction to $\partial\Delta_1^1$ to $\Delta_1^1 := \{\langle s,t\rangle : 2 \times 2 \mid s = t\}$.

$$\sum_{h:\left\langle \Delta^1 \to A\middle|^{\partial\Delta^1}_{[x,z]}\right\rangle} \left\langle \Delta^2 \to A\middle|^{\partial\Delta^2}_{[x,y,z,f,g,h]}\right\rangle \simeq \sum_{l:\left\langle \partial\Delta^2 \to A\middle|^{\Lambda_1^2}_{[x,y,z,f,g]}\right\rangle} \left\langle \Delta^2 \to A\middle|^{\partial\Delta^2}_{l}\right\rangle \simeq$$

$$\left\langle \Delta^2 \to A\middle|^{\Lambda_1^2}_{[x,y,z,f,g]}\right\rangle.$$ Using 5.4, $(\Delta^2 \to A) \simeq \sum_{k:\Lambda_1^2 \to A} \left\langle \Delta^2 \to A\middle|^{\Lambda_1^2}_{k}\right\rangle$.

Since the projection from a total space is an equivalence exactly when all the fibers are contractible, the result follows.

Segal types as ∞-categories

Corollary 6.5

If X is either a type or a shape and $A : X \to \mathcal{U}$ is such that each $A(x)$ is a Segal type for all $x : X$, then the dependent function type $\prod_{x:X} A(x)$ is a Segal type.

Segal types as ∞-categories

Corollary 6.5

If X is either a type or a shape and $A : X \to \mathcal{U}$ is such that each $A(x)$ is a Segal type for all $x : X$, then the dependent function type $\prod_{x:X} A(x)$ is a Segal type.

Proof. By the rearrangement of function types, we have $(\Delta^2 \to \prod_{x:X} A(x)) \simeq \prod_{x:X}(\Delta^2 \to A(x))$ and similarly for Λ_1^2. Finally, relative function extensionality axiom and the theorem 6.4 does the trick.

Segal types as ∞-categories

Corollary 6.5

If X is either a type or a shape and $A : X \to \mathcal{U}$ is such that each $A(x)$ is a Segal type for all $x : X$, then the dependent function type $\prod_{x:X} A(x)$ is a Segal type.

Proof. By the rearrangement of function types, we have $(\Delta^2 \to \prod_{x:X} A(x)) \simeq \prod_{x:X}(\Delta^2 \to A(x))$ and similarly for Λ_1^2. Finally, relative function extensionality axiom and the theorem 6.4 does the trick.

Definition 6.6 (Identity)

For any $x : A$, define a term $\mathbf{id}_x : \hom_A(x, x)$ by $\mathrm{id}_x(s) \equiv x$ for all $s : 2$.

Segal types as ∞-categories

Corollary 6.5

If X is either a type or a shape and $A : X \to \mathcal{U}$ is such that each $A(x)$ is a Segal type for all $x : X$, then the dependent function type $\prod_{x:X} A(x)$ is a Segal type.

Proof. By the rearrangement of function types, we have $(\Delta^2 \to \prod_{x:X} A(x)) \simeq \prod_{x:X}(\Delta^2 \to A(x))$ and similarly for Λ_1^2. Finally, relative function extensionality axiom and the theorem 6.4 does the trick.

Definition 6.6 (Identity)

For any $x : A$, define a term $\mathbf{id}_x : \hom_A(x, x)$ by $\mathrm{id}_x(s) \equiv x$ for all $s : 2$.

Proposition 6.7 (Unit law)

If A is a Segal type with terms $x, y : A$, then for any $f : \hom_A(x, y)$ we have $\mathrm{id}_y \circ f = f$ and $f \circ \mathrm{id}_x = f$.

Segal types as ∞-categories

Proof. For any $f : \hom_A(x, y)$ we have a canonical 2-simplex:

$$\lambda s, t. f(s) : \left(\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd} \right)$$

Segal types as ∞-categories

Proof. For any $f : \hom_A(x,y)$ we have a canonical 2-simplex:

$$\lambda s,t.f(s) : \left(\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd} \right)$$

To check that this has the right boundary, we see that $(s,0) \mapsto f(s)$ and $(s,s) \mapsto f(s)$, while $(1,t) \mapsto f(1) = y$. Thus, by uniqueness of composites, $\mathrm{id}_y \circ f = f$. Similarly, $f \circ \mathrm{id}_x = f$.

Segal types as ∞-categories

Proof. For any $f : \hom_A(x, y)$ we have a canonical 2-simplex:

$$\lambda s, t. f(s) : \left(\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd} \right)$$

To check that this has the right boundary, we see that $(s, 0) \mapsto f(s)$ and $(s, s) \mapsto f(s)$, while $(1, t) \mapsto f(1) = y$. Thus, by uniqueness of composites, $\mathrm{id}_y \circ f = f$. Similarly, $f \circ \mathrm{id}_x = f$.

Proposition 6.8 (Associativity)

If A is Segal type with terms $x, y, z, w : A$, then for any $f : \hom_A(x, y)$, $g : \hom_A(y, z)$, $h : \hom_A(z, w)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Segal types as ∞-categories

Proof. For any $f : \hom_A(x,y)$ we have a canonical 2-simplex:

$$\lambda s,t.f(s) : \left(\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd} \right)$$

To check that this has the right boundary, we see that $(s,0) \mapsto f(s)$ and $(s,s) \mapsto f(s)$, while $(1,t) \mapsto f(1) = y$. Thus, by uniqueness of composites, $\mathrm{id}_y \circ f = f$. Similarly, $f \circ \mathrm{id}_x = f$.

Proposition 6.8 (Associativity)

If A is Segal type with terms $x,y,z,w : A$, then for any $f : \hom_A(x,y)$, $g : \hom_A(y,z)$, $h : \hom_A(z,w)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Proof. The type $\displaystyle\sum_{p:\hom_{A^2}(f,h)} \hom^2_{A^2} \left(\begin{tikzcd} & g \arrow[dr, "\mathrm{comp}_{h,g}"] & \\ f \arrow[ur, "\mathrm{comp}_{g,f}"] \arrow[rr, "p"'] & & h \end{tikzcd} \right)$ is contractible,

hence inhabited.

Segal types as ∞-categories

$$
\begin{tikzcd}
& y \arrow[dd, "g"] \arrow[dr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[dr, "g\circ f"'] \arrow[r, phantom, "\mathrm{comp}"] & {} \arrow[r, phantom, "\mathrm{comp}"] & w \\
& z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& y \arrow[dr, "g"] \arrow[dd] & \\
x \arrow[ur, "f"] \arrow[rr, "g"] \arrow[dd, "f"'] \arrow[rrdd, dashed, bend left, "m'"] & & z \arrow[dd, "h"] \\
& z \arrow[dr] & \\
y \arrow[ur, "g"] \arrow[rr, "h"] & & w
\end{tikzcd}
\qquad
\begin{tikzcd}
& y \arrow[dr, "h\circ g"] \arrow[d, phantom, "s"] & \\
x \arrow[ur, "f"] \arrow[rr, "m'"'] \arrow[dr, "g\circ f"'] & {} \arrow[d, phantom, "r'"] & w \\
& z \arrow[ur, "h"'] &
\end{tikzcd}
$$

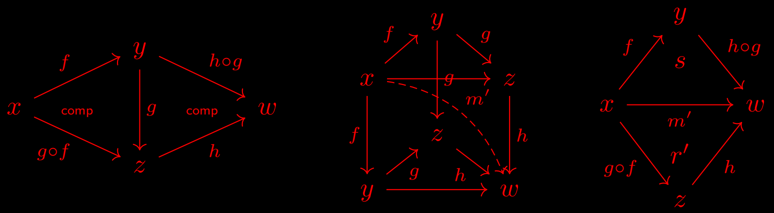
The second component of the inhabitant is a 2-simplex witness $\Delta^2 \times 2 \to A$.

Segal types as ∞-categories

$$\begin{tikzcd}
 & y \arrow[dd, "g"] \arrow[dr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[dr, "g\circ f"'] \arrow[r, phantom, "\text{comp}"] & {} \arrow[r, phantom, "\text{comp}"] & w \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[dd, "g"] & \\
x \arrow[ur, "f"] \arrow[rr] \arrow[dd, "f"'] \arrow[ddrr, dashed, bend left, "m'"] & & z \arrow[dd, "h"] \\
 & z \arrow[dr] & \\
y \arrow[ur, "g"] \arrow[rr, "h"] & & w
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \arrow[dr, "h\circ g"] \arrow[d, phantom, "s"] & \\
x \arrow[ur, "f"] \arrow[rr, "m'"'] \arrow[dr, "g\circ f"'] & {} & w \\
 & z \arrow[ur, "h"'] \arrow[u, phantom, "r'"] &
\end{tikzcd}$$

The second component of the inhabitant is a 2-simplex witness $\Delta^2 \times \mathbb{2} \to A$.
Now think of the function $\lambda(t_1, t_2, t_3).((t_1, t_3), t_2) : \Delta^3 \to \Delta^2 \times \mathbb{2}$ picking out the middle shuffle of the prism. Identifying the faces with restrictions,
$\lambda(s, t).(s, s, t) : \Delta^2 \to \Delta^3$ and $\lambda(s, t).(s, t, t) : \Delta^2 \to \Delta^3$, with the common edge $\lambda t.(t, t, t) : \Delta^1 \to \Delta^3$.

Segal types as ∞-categories



The second component of the inhabitant is a 2-simplex witness $\Delta^2 \times \mathbb{2} \to A$.
Now think of the function $\lambda(t_1,t_2,t_3).((t_1,t_3),t_2) : \Delta^3 \to \Delta^2 \times \mathbb{2}$ picking out the middle shuffle of the prism. Identifying the faces with restrictions, $\lambda(s,t).(s,s,t) : \Delta^2 \to \Delta^3$ and $\lambda(s,t).(s,t,t) : \Delta^2 \to \Delta^3$, with the common edge $\lambda t.(t,t,t) : \Delta^1 \to \Delta^3$.
This edge defines an inhabitant $m' : \hom_A(x,w)$, while the pair of 2-simplices define witnesses that m' is the composite of $h \circ g$ and f, and that m' is the composite of h and $g \circ f$, respectively. In particular, $(h \circ g) \circ f = h \circ (g \circ f)$.

Segal types as ∞-categories

Proposition 6.9

For any $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$ and $h : \hom_A(x, z)$ in a Segal type A, the natural map

$$(g \circ f = h) \to \hom^2_A \left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & y \end{tikzcd} \right)$$

is an equivalence.

Segal types as ∞-categories

Proposition 6.9

For any $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$ and $h : \hom_A(x, z)$ in a Segal type A, the natural map

$$(g \circ f = h) \to \hom_A^2 \left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & y \end{tikzcd} \right)$$

is an equivalence.

Proof. The map is given by path induction, since when $h \equiv g \circ f$ the codomain is inhabited by $\mathsf{comp}_{g,f}$. Now we just sum over h and show that both the total spaces are contractible.

Segal types as ∞-categories

Proposition 6.9

For any $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$ and $h : \hom_A(x, z)$ in a Segal type A, the natural map

$$(g \circ f = h) \to \hom_A^2 \left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & y \end{tikzcd} \right)$$

is an equivalence.

Proof. The map is given by path induction, since when $h \equiv g \circ f$ the codomain is inhabited by $\mathsf{comp}_{g,f}$. Now we just sum over h and show that both the total spaces are contractible.

The homotopies between arrows in a Segal type behave like a 2-category up to homotopy. For instance, given $p : f =_{\hom_A(x,y)} g$ and $q : g =_{\hom_A(x,y)} h$, we can vertically compose them to get $p \cdot q : f =_{\hom_A(x,y)} h$.

Segal types as ∞-categories

Proposition 6.10 (Horizontal composition)

Given $p : f =_{\hom_A(x,y)} g$ and $q : h =_{\hom_A(y,z)} k$ in a Segal type A, there is a concatenated equality $q \circ_2 p : h \circ f =_{\hom_A(x,z)} k \circ g$.

Segal types as ∞-categories

Proposition 6.10 (Horizontal composition)

Given $p : f =_{\hom_A(x,y)} g$ and $q : h =_{\hom_A(y,z)} k$ in a Segal type A, there is a concatenated equality $q \circ_2 p : h \circ f =_{\hom_A(x,z)} k \circ g$.

Proposition 6.11 (Whiskering)

Given $p : f =_{\hom_A(x,y)} g$ and $h : \hom_A(y,z)$ and $k : \hom_A(w,x)$ in a Segal type A, we have

$$\mathsf{refl}_h \circ_2 p = \mathsf{ap}_{(h \circ \_)}(p)$$
$$p \circ_2 \mathsf{refl}_k = \mathsf{ap}_{(\_ \circ k)}(p).$$

Segal types as ∞-categories

Proposition 6.10 (Horizontal composition)

Given $p : f =_{\hom_A(x,y)} g$ and $q : h =_{\hom_A(y,z)} k$ in a Segal type A, there is a concatenated equality $q \circ_2 p : h \circ f =_{\hom_A(x,z)} k \circ g$.

Proposition 6.11 (Whiskering)

Given $p : f =_{\hom_A(x,y)} g$ and $h : \hom_A(y,z)$ and $k : \hom_A(w,x)$ in a Segal type A, we have

$$\begin{gathered} \mathsf{refl}_h \circ_2 p = \mathsf{ap}_{(h \circ \_)}(p) \\ p \circ_2 \mathsf{refl}_k = \mathsf{ap}_{(\_ \circ k)}(p). \end{gathered}$$

Proposition 6.12

We have the following equality in a Segal type whenever it makes sense:

$$(q' \cdot p') \circ_2 (q \cdot p) = (q' \circ_2 q) \cdot (p' \circ_2 p).$$

Segal types as ∞-categories

Proposition 6.10 (Horizontal composition)

Given $p : f =_{\hom_A(x,y)} g$ and $q : h =_{\hom_A(y,z)} k$ in a Segal type A, there is a concatenated equality $q \circ_2 p : h \circ f =_{\hom_A(x,z)} k \circ g$.

Proposition 6.11 (Whiskering)

Given $p : f =_{\hom_A(x,y)} g$ and $h : \hom_A(y,z)$ and $k : \hom_A(w,x)$ in a Segal type A, we have

$$\mathsf{refl}_h \circ_2 p = \mathsf{ap}_{(h\circ\_)}(p)$$
$$p \circ_2 \mathsf{refl}_k = \mathsf{ap}_{(\_\circ k)}(p).$$

Proposition 6.12

We have the following equality in a Segal type whenever it makes sense:

$$(q' \cdot p') \circ_2 (q \cdot p) = (q' \circ_2 q) \cdot (p' \circ_2 p).$$

Proof. Path induction.

Segal types as ∞-categories

Proposition 6.13 (Fillings of 3-dimensional horns)

In a Segal type A, suppose given arrows f, g, h, k, l, m and equalities

$$p : g \circ f =_{\hom_A(x,z)} k \quad q : h \circ g =_{\hom_A(z,w)} l \quad r : h \circ k =_{\hom_A(x,w)} m$$

corresponding to 2-simplices that fill out the following horn $\Lambda_2^3 \to A$:

$$\begin{tikzcd}
 & y \arrow[dd, "g"] \arrow[dr, "l"] & \\
x \arrow[ur, "f"] \arrow[dr, "k"'] \arrow[r, phantom, "p"] & & w \arrow[l, phantom, "q"] \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \arrow[dr, "l"] \arrow[d, phantom, "-"] & \\
x \arrow[ur, "f"] \arrow[rr, "m"'] \arrow[dr, "k"'] & {} & w \\
 & z \arrow[ur, "h"'] \arrow[u, phantom, "r"] &
\end{tikzcd}$$

Then the horn has a filler $\Delta^3 \to A$ corresponding to the concatenated equality

$$l \circ f \overset{q}{=} (h \circ g) \circ f = h \circ (g \circ f) \overset{p}{=} h \circ k \overset{r}{=} m.$$

where p and q are whiskered by h and f respectively.

Segal types as ∞-categories

Proof. Trick is to do path induction on p and q. Then the 2-simplices corresponding to p and q are now $\mathsf{comp}_{g,f}$ and $\mathsf{comp}_{h,g}$ while the above chain of equalities reduces to $(h \circ g) \circ f = h \circ (g \circ f) \overset{\mathsf{r}}{=} m$.

Segal types as ∞-categories

Proof. Trick is to do path induction on p and q. Then the 2-simplices corresponding to p and q are now $\mathsf{comp}_{g,f}$ and $\mathsf{comp}_{h,g}$ while the above chain of equalities reduces to $(h \circ g) \circ f = h \circ (g \circ f) \overset{\mathsf{r}}{=} m$. By contractibility, we have $(m', r') = (m, r)$. Also we have the decomposition of 3-2-horns,

$$(\Lambda_2^3 \to A) \simeq \sum_{\alpha : \Delta^2 \cup_{\Delta^1} \Delta^2 \to A} \left\langle \Delta^2 \to A \Big|_{\alpha}^{\Lambda_1^2} \right\rangle$$

where $\Delta^2 \cup_{\Delta^1} \Delta^2$ denotes the pushout of $\mathsf{comp}_{g,f}$ and $\mathsf{comp}_{h,g}$, with Λ_1^2 being $g \circ f$ and h. Thus, equality $(m', r') = (m, r)$ in $\left\langle \Delta^2 \to A \Big|_{[h, g\circ f]}^{\Lambda_1^2} \right\rangle$ yields $[\mathsf{comp}_{g,f}, \mathsf{comp}_{h,g}, r'] = [\mathsf{comp}_{g,f}, \mathsf{comp}_{h,g}, r]$.

Segal types as ∞-categories

Proof. Trick is to do path induction on p and q. Then the 2-simplices corresponding to p and q are now $\mathsf{comp}_{g,f}$ and $\mathsf{comp}_{h,g}$ while the above chain of equalities reduces to $(h \circ g) \circ f = h \circ (g \circ f) \overset{\mathsf{r}}{=} m$. By contractibility, we have $(m', r') = (m, r)$. Also we have the decomposition of 3-2-horns,

$$(\Lambda_2^3 \to A) \simeq \sum_{\alpha : \Delta^2 \cup_{\Delta^1} \Delta^2 \to A} \left\langle \Delta^2 \to A \Big|_{\alpha}^{\Lambda_1^2} \right\rangle$$

where $\Delta^2 \cup_{\Delta^1} \Delta^2$ denotes the pushout of $\mathsf{comp}_{g,f}$ and $\mathsf{comp}_{h,g}$, with Λ_1^2 being $g \circ f$ and h. Thus, equality $(m', r') = (m, r)$ in $\left\langle \Delta^2 \to A \Big|_{[h, g\circ f]}^{\Lambda_1^2} \right\rangle$ yields $[\mathsf{comp}_{g,f}, \mathsf{comp}_{h,g}, r'] = [\mathsf{comp}_{g,f}, \mathsf{comp}_{h,g}, r]$.

The 3-simplex in 6.8 is a term of $\left\langle \Delta^3 \to A \Big|_{[\mathsf{comp}_{g,f}, \mathsf{comp}_{h,g}, r']}^{\Lambda_2^3} \right\rangle$, we transport this term across the equality to get a term of $\left\langle \Delta^3 \to A \Big|_{[\mathsf{comp}_{g,f}, \mathsf{comp}_{h,g}, r]}^{\Lambda_2^3} \right\rangle$, our desired 3-simplex.

Thank you for being an attentive and morphically coherent audience, as your participation enriches the categorical landscape of this presentation.