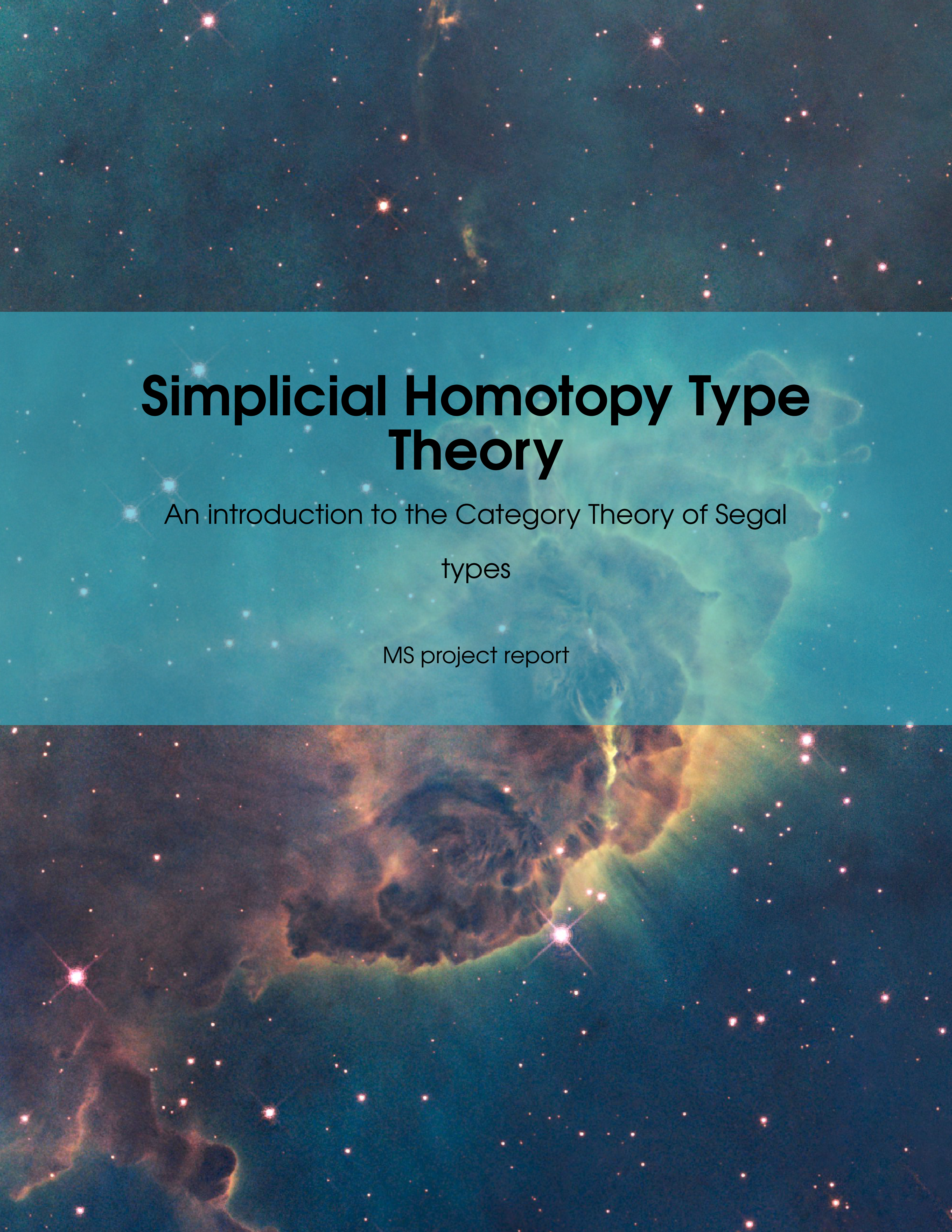

Simplicial Homotopy Type Theory

An introduction to the Category Theory of Segal types

MS project report

SIMPLICIAL HOMOTOPY TYPE THEORY

An introduction to the category theory of Segal types

A project report

by

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Declaration

I hereby declare that the work in this project report has been carried out by me in the Integrated Ph.D. program under the supervision of Professor Siddhartha Gadgil in the partial fulfillment of the requirements of the Master of Science degree of the Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, or any other title elsewhere.

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June, 2023.

Professor Siddhartha Gadgil
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Abstract

In this project, we first go through a brief introduction to ∞-categories, followed by some high-level background in homotopy type theory (HoTT). Next, we propose foundations for a synthetic theory of (∞,1)-categories within homotopy type theory. We axiomatize a directed interval type, then define higher simplices from it and use them to probe the internal categorical structures of arbitrary types. We introduce the general theory of shapes and then add to it the theory of a strict interval. Furthermore, we will see a bunch of important equivalences involving extension types. Finally we define *Segal types* or synthetic pre-(∞,1)-categories, in which binary composites exist uniquely up to homotopy and this automatically ensures composition is coherently associative and unital at all dimensions.

To make the bookkeeping in such proofs manageable, we use a three-layered type theory with shapes, whose contexts are extended by polytopes within directed cubes, which can be abstracted over using "extension types" that generalize the path-types of cubical type theory.

Contents

1. Introduction to 1,2,3,...∞-categories

We think of an ordinary 1-category as a linguistic template for a mathematical theory with nouns and verbs. But as the objects mathematicians study gets more complicated, we need a more robust linguistic template with adjectives, adverbs and so on, and that is when the idea of a higher category or an ∞-category emerges. So, mathematicians would definitely like to use ∞-categories to help systemize their work in the same way they are using ordinary categories today.

In addition to the objects and arrows in an ordinary 1-category, which we think of as 0 and 1-dimensional respectively, we can visualize or think of an ∞-category as having arrows between arrows, which we think of as inhabiting the 2-dimensional space. Then we furher have arrows between, arrows between arrows inhabiting the 3-dimensional space and so on.

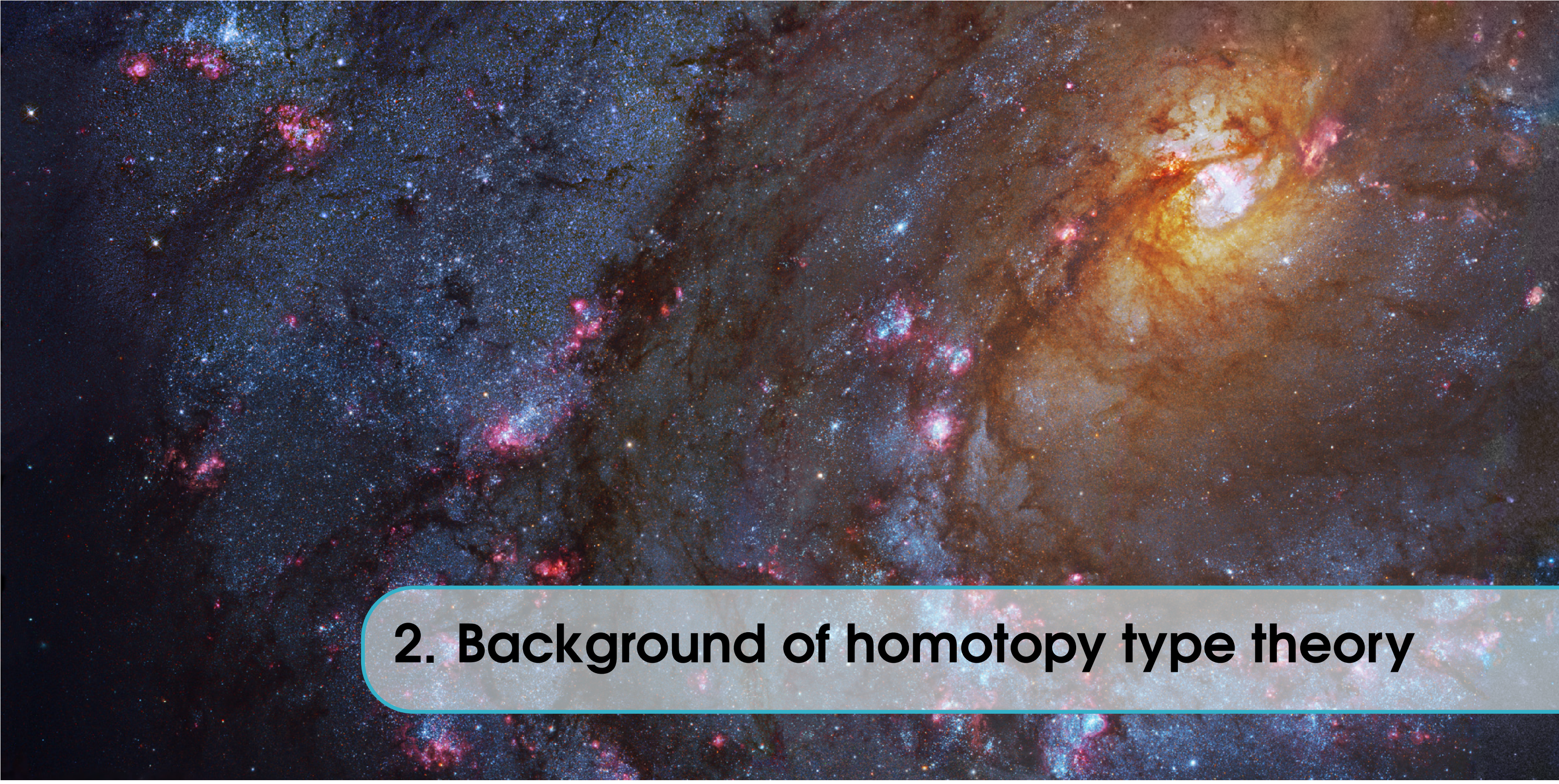
Once it was realized that higher groupoids should correspond to homotopy types, and that for a wide range of applications it is sufficient to assume that all "higher morphisms" are invertible, a number of different definitions of these so-called $(\infty, 1)$-categories was suggested, with the index "1" referring to the fact that the morphisms above the lowest dimension are weakly invertible. Here the homotopies defining the higher morphisms of an ∞-category are to be regarded as data rather than as mere witnesses to an equivalence relation borne by the 1-dimensional morphisms. This shift in perspective is illustrated by the relationship between two algebraic invariants of a topological space: the fundamental groupoid Π_1, an ordinary 1-category, and the fundamental ∞-groupoid Π_∞, an ∞-category in which all of the morphisms are weakly invertible. The objects in both the categories are the points of the ambient topological space, but in the former, the 1-morphisms are homotopy classes of paths, while in the latter, the 1-morphisms are the paths themselves and the 2-morphisms are explicit endpoint preserving homotopies. The groupoid $\Pi_1(X)$ has a very nice homotopic property, that is, it retains information on $\pi_0(X)$ and $\pi_1(X)$ of X but forgets all higher homotopy groups of X. And the reason is, we took homotopy classes of paths as arrows. Could we have taken instead topological spaces of paths, we would have chance to retain all information about the homotopy type of X, and that is why we thought of having an upgraded version, $\Pi_\infty(X)$. This is, however, easier said than done. It is easy to compose homotopy classes of paths, but there is no canonical way of composing the actual paths. To encompass examples such as these, all of the categorical structures in an ∞-category are weak. Even at the base level of 1-morphisms, composition is not necessarily uniquely defined but is instead witnessed by a 2-morphism and associative up to a 3-morphism whose boundary data involves specified 2-morphism witnesses. Thus, ∞-category valued diagrams cannot be said to commute on the nose but are instead interpreted as homotopy coherent, with explicitly specified higher homotopies.

Let us look at a more algebraic example. An important notion of homological algebra is the notion of derived functor. Study of derived functors led to the notion of derived category in which the standard ambiguity in the choice of resolutions used to calculate derived functors, disappears. As a first step, one constructs the category of complexes $C(\mathscr{A})$, $\mathscr{A}$ being an abelian category where all projective resolutions, their images after application of functors, live. The second step is similar to the notion of localization of a ring: given a ring R and a multiplicatively closed subset S of its elements, one defines a ring homomorphism $R \to R[S^{-1}]$ such that the image of each element in S becomes invertible in $R[S^{-1}]$ and universal for this property. Localizing $C(\mathscr{A})$ with respect to the collection of quasi-isomorphisms, we get $D(\mathscr{A})$, the derived category of $\mathscr{A}$. The construction of derived category $D(\mathscr{A})$ is a close relative to the construction of the homotopy category of topological spaces, when we factor the set of continuous maps by an equivalence relation. The notion of derived category is not very convenient, approximately for the same reasons we already mentioned. Localizing a category, we destroy an important information, similarly to destroying information about higher homotopy groups of X in the construction of the fundamental groupoid $\Pi_1(X)$. So, one would like an upgraded version of derived category. In addition, there is a pleasant "side effect" in replacing the derived category $D(\mathscr{A})$ with an infinity category. As it is well-known, $D(\mathscr{A})$ is a triangulated category, that is an additive category endowed with a shift endofunctor, with a chosen collection of diagrams called distinguished triangles, satisfying a list of properties. The notion of triangulated category is a very important, but very unnatural one. Fortunately, the respective infinity categorical notion is very natural, a property of infinity category called stability rather than a collection of extra structures like shift functor and distinguished triangles.

The abundance of competing definitions of $(\infty, 1)$-category is somewhat similar to the abundance of programming languages or of models of computation where all of them have in mind the same idea of computability, but realize this idea differently. A fundamental challenge in giving a model independent definition of ∞-categories has to do with giving a precise mathematical meaning of the notion of a weak composition law, not just for the 1-morphisms but also for the morphisms in higher dimensions. As a result, While proving theorems about ∞-categories, we need to first pick a specific definition, like choosing coordinates and prove theorems with reference to that definition, thereby providing a translation problem. It is less obvious how to compare different formalizations of ∞-categories. In all existing approaches, $(\infty, 1)$-categories are realized as fibrant-cofibrant objects in a certain Quillen model category for example, simpilicial sets with Joyal model structure, bisimplicial sets with Segal category or complete Segal model structure, simplicial categories or simplicially enriched categories, etc. The graph of Quillen equivalences between different model categories in the above list is connected, which implies that at least the homotopy categories of different versions of $(\infty, 1)$-categories are equivalent. This, however, seems too weak at first sight. On the other hand, if we choose our favorite definition of ∞-category, we can see that, first of all, any model category gives rise to an ∞-category, the underlying ∞-category, and, furthermore, Quillen equivalent model categories give rise to equivalent ∞-categories. In particular, ∞-categories underlying different models of infinity categories, are equivalent. For practical, aesthetic, and moral reasons, the ultimate desire of practitioners is to work "model independently", meaning that theorems proven with any of the models of $(\infty, 1)$-categories would apply to them all, with the technical details inherent to any particular model never entering the discussion. Since all models of $(\infty, 1)$-categories "have the same homotopy theory", the general consensus is that the choice of model should not matter greatly, but one obstacle to proving results of this kind is that, to a large extent, precise versions of the categorical definitions that have been established for quasi-categories had not been given for the other models. In cases where comparable definitions do exist in different models, an ad hoc heuristic proof of model invariance of the categorical notion in question can typically be supplied, with details to be filled in by experts fluent in the combinatorics of each model, but it would be more reassuring to have a systematic method of comparing the category theory of $(\infty, 1)$-categories in different models via arguments that are somewhat closer to the ground.

Recently, Emily Riehl and Dominic Verity has come up with an unique approach where an ∞-category is thought of living in their own category, the category of all ∞-categories, the ∞-cosmos. The strategy behind this is to prove theorems about ∞-cosmos in general and then they will specialize to give the theorems about ∞-categories proven in absolutely every model, that apply universally.

However, our goal is to give a model independent definition of an $(\infty,1)$-category. We propose foundations for a synthetic theory of $(\infty,1)$-categories within homotopy type theory (HoTT) in the subsequent chapters.



2. Background of homotopy type theory

2.1 What is homotopy type theory?

Homotopy type theory (HoTT) is a relatively new area of mathematics that seeks to connect two seemingly unrelated topics: homotopy theory and type theory. Homotopy theory is a branch of mathematics that studies the properties of topological spaces and their invariants under continuous deformations, known as homotopies. It studies the notion of equivalence between geometric objects, and how they can be transformed or deformed into each other while preserving certain structural features. On the other hand, type theory is a branch of mathematical logic that studies the formalization of mathematical objects and their relationships. It provides a framework for classifying mathematical objects into different types based on their properties and relationships, and it allows for precise reasoning about the behavior and interaction of these objects. HoTT offers a fresh perspective on the foundations of mathematics, and the potential to provide new insights into the nature of mathematical objects. HoTT allows us to combine these two fields by defining types as spaces and taking advantage of the concept of homotopy, which captures the notion of continuous deformation in topology. This allows us to reason about types in a more geometric way and take advantage of homotopical tools to investigate them.

We should stress that these "spaces" are treated purely homotopically, not topologically. For instance, there is no notion of "open subset" of a type or of "convergence" of a sequence of terms of a type. We only have "homotopical" notions, such as paths between points and homotopies between paths, which also make sense in other models of homotopy theory such as simplicial sets. Thus, it would be more accurate to say that we treat types as ∞-groupoids, this is a name for the "invariant objects" of homotopy theory which can be presented by topological spaces, simplicial sets, or any other model for homotopy theory. However, it is convenient to sometimes use topological words such as "space" and "path", as long as we remember that other topological concepts are not applicable.

The key new idea of the homotopy interpretation is that the logical notion of identity $a = b$ of two objects $a, b : A$ of the same type A can be understood as the existence of a path $p : a \rightsquigarrow b$ from point a to point b in the space A. This also means that two functions $f,g : A \to B$ can be identified if they are homotopic, since a homotopy is just a family of paths $p_x : f(x) \rightsquigarrow g(x)$ in B, one for each $x : A$. In type theory, for every type A there is a type Id_A or $=_A$ of identifications of two terms of A. In homotopy type theory, this is just the path space A^I of all continuous maps $I \to A$ from the unit interval I. In this way, a term $p : a = b$ represents a path $p : a \rightsquigarrow b$ in A.

The idea of homotopy type theory arose around 2006 in independent work by Awodey and Warren and Voevodsky, but it was inspired by Hofmann and Streicher's earlier groupoid interpretation, in their disproof of the uniqueness of identity proofs (UIP) using groupoids. Intuitively, the identity type over a groupoid G is interpreted as the discrete groupoid $Arr(G)$ of arrows in G, so that an identity witness $f : a =_A b$ becomes an arrow $f : a \to b$ in G. However, unlike in type theory, these cannot in turn be further related by identity terms of higher type $f =_{(a =_A b)} g$, since a conventional groupoid generally has no such higher-dimensional structure. Thus the groupoid semantics validates a certain truncation principle, stating that all higher identity types are trivial, a form of extensionality one dimension up. The groupoid laws for the identity types are strictly satisfied in these models, rather than holding only up to propositional equality. This situation suggests the use of the higher-dimensional analogues of groupoids, as formulated in homotopy theory, in order to provide models admitting non-trivial higher identity types. In homotopy type theory, each type A can be seen to have the structure of an ∞-groupoid. We can iterate the identity type: we can form the type $p =_{(x =_A y)} q$ of identifications between identifications p,q, and the type $r =_{(p =_{(x =_A y)} q)} s$, and so on. The structure of this tower of identity types corresponds precisely to that of the continuous paths and higher dimensional homotopies between them in a space, or an ∞-groupoid. As a result, homotopy type theory can be viewed as a synthetic theory of ∞-groupoids in different homotopy theoretic models. One such classic model is the Voevodsky's simplicial set model in which types are regarded as Kan complexes in the Quillen model structure. In this interpretation, the identity type $a =_K b$ of any two points a,b in a Kan complex K is itself a Kan complex.

Voevodsky recognized that the simplicial interpretation of type theory satisfies a further crucial property, dubbed *univalence*, which had not previously been considered much in type theory. Adding univalence to type theory in the form of a new axiom has far-reaching consequences, many of which are natural, simplifying and compelling. The univalence axiom also further strengthens the homotopical view of type theory, since it holds in the simplicial model and other related models, while failing under the view of types as just discrete sets.

2.2 Mystery of identity types: the Univalence axiom

One feature of dependent type theory which has previously remained comparatively unexploited, however, is its richer treatment of equality. In traditional foundations like set theory, equality carries no information beyond its truth-value: if two things are equal, they are equal in at most one way. This is fine for equality between elements of discrete sets, but it is unnatural for objects of categories, or points of spaces. In particular, it is at odds with the informal mathematical practice of treating isomorphic objects as equal. As a reason, this usage must be so often disclaimed as an abuse of language, and kept rigorously away from formal statements, even though it is so appealing.

In dependent type theory, equalities can carry information: two things may be equal in multiple ways. So the basic objects—the *types*—may behave not just like discrete sets, but more generally like higher groupoids, or spaces. And, crucially, this is the only equality one can talk about within the logical system: one cannot ask whether elements of a type are "equal on the nose", in the classical sense. The logical language only allows one to talk about properties and constructions which respect its equality.

The *Univalence Axiom*, introduced by Vladimir Voevodsky, strengthens this characteristic. In classical foundations one has sets of sets, or classes of sets, and uses these to quantify over classes of structures. Similarly, in type theory, types of types—*universes*—are a key feature of the language. Very briefly, the basic idea of the univalence axiom can be explained as follows. In type theory, one can have a universe $\mathscr{U}$, the terms of which are themselves types, $A : \mathscr{U}$. Those types that are terms of $\mathscr{U}$ are commonly called small types. Like any type, $\mathscr{U}$ has an identity type $\mathrm{Id}_{\mathscr{U}}$, which expresses the identity relation $A = B$ between small types. Thinking of types as spaces, $\mathscr{U}$ is a space, the points of which are spaces themselves. To understand its identity type, we must ask, how to interpret a path $p : A \rightsquigarrow B$ between spaces in $\mathscr{U}$? The univalence axiom says that such paths correspond roughly to homotopy equivalences $A \simeq B$. A bit more precisely, given any small types A and B, in addition to the primitive type $A = B$ of identifications of A with B, there is the defined type $A \simeq B$ of equivalences from A to B. Since the identity map on any object is an equivalence, by path induction, there is a canonical map, $(A = B) \to (A \simeq B)$. The univalence axiom states that this map is itself an equivalence, i.e., $(A = B) \simeq (A \simeq B)$.

The Univalence Axiom states that equality between types, as elements of a universe, is the same as equivalence between them, as types. It formalises the practice of treating equivalent structures as completely interchangeable. It ensures that one can only talk about properties of types, or more general structures, that respect such equivalence. It helps solidify the idea of types as some kind of spaces, in the homotopy-theoretic sense. From the homotopical point of view, univalence implies that spaces of the same homotopy type are connected by a path in the universe $\mathscr{U}$, in accord with the intuition of a classifying space for (small) spaces. From the logical point of view, however, it is a radically new idea: it says that isomorphic things can be identified! Mathematicians are of course used to identifying isomorphic structures in practice, but they generally do so by "abuse of notation", or some other informal device, knowing that the objects involved are not "really" identical. But in this new foundational scheme, such structures can be formally identified, in the logical sense that every property or construction involving one also applies to the other. Indeed, the identification is now made explicit, and properties and constructions can be systematically transported along it.

2.3 HoTT and Higher categories

As we discussed in section 2.1 that homotopy type theory serves as a foundation for the theory of ∞-groupoids, we are curious to know whether homotopy type theory can be viewed as a foundational system for higher categories in general. Interpreting types directly as higher categories runs into various problems, such as the fact that not all maps between categories are exponentiable so that not all Π-types exist, and that there are numerous different kinds of "fibrations" given the various possible functorialities and dimensions of categories appearing as fibers. There is no reason in principle to think these problems insurmountable, and many possible solutions have been proposed. However, in this project we learn to pursue a somewhat indirect route to a synthetic theory of higher categories, which has its own advantages, and may help illuminate some aspects of what an eventual more direct theory might look like.

Homotopy type theory admits semantics not only in simplicial sets, but in many other model categories. In particular, as shown by Shulman, it can be interpreted in the Reedy model structure on bisimplicial sets, also called simplicial spaces. This model structure, in turn, admits a left Bousfield localization called the complete Segal space model structure, which presents the homotopy theory and indeed also the category theory of $(\infty, 1)$-categories. We cannot interpret homotopy type theory in its usual form, in the complete Segal space model structure directly due to its lack of right properness among other things, but we can interpret it in the Reedy model structure and identify internally some types that correspond to Segal spaces and complete Segal spaces. That is, in contrast to ordinary homotopy type theory where the basic objects are exactly the "synthetic ∞-groupoids", in our theory the basic objects are something more general, inside of which we identify two classes that we regard as "synthetic pre-$(\infty, 1)$-categories" and "synthetic $(\infty, 1)$-categories".

The identification of these "category-like types", and the study of their properties, depends on adding certain structure to homotopy type theory that is characteristic of the bisimplicial set model. The fundamental such structure is a "directed interval" type, which (thinking categorically) we denote $\mathbb{2}$. As it does in ordinary category theory, the directed interval $\mathbb{2}$ detects arrows representably i.e., for any type A the function type $\mathbb{2} \to A$ is the "type of arrows in A". The directed interval $\mathbb{2}$ possesses a lot of useful structure, like the internal incarnation of this structure which we will discuss in chapter 5, which is what is visible in the homotopy type theory of bisimplicial sets, is nicely summarized by saying that it is a strict interval: a totally ordered set with distinct bottom and top elements called 0 and 1 respectively. The strict interval structure on $\mathbb{2}$ i.e., Δ^1 placed in the categorical direction (see definition 3.3.1) allows us to define the higher simplices from it internally, and hence the higher categorical structure of types. For instance, $\Delta^2 = \{(s,t) : \mathbb{2} \times \mathbb{2} \mid t \leqslant s\}$. We regard a map $\alpha : \Delta^2 \to A$ as a "commutative triangle" in A witnessing that the composite of $\lambda t.\alpha(t,0) : \Delta^1 \to A$ and $\lambda t.\alpha(1,t) : \Delta^1 \to A$ is $\lambda t.\alpha(t,t) : \Delta^1 \to A$. Importantly, for a general type A, two given composable arrows i.e., two functions $f, g : \mathbb{2} \to A$ with $f(1) = g(0)$, may not have any such "composite", or they may have more than one. If any two composable arrows have a unique composite in the homotopical sense, that the type of such composites with their witnesses is contractible, we call A a Segal type or a synthetic pre-$(\infty,1)$-category. If a Segal type satisfies a further condition analogous to Rezk's "completeness" condition for Segal spaces, we call it a Rezk type or synthetic $(\infty,1)$-category. Our further goal in this project is to study the basic category theory of Segal types.

Given that $\mathbb{2} \to A$ represent the arrow type of A, we would want to talk about the dependent type of arrows, i.e., given two terms $x, y : A$, the "type of arrows from x to y", i.e., the type of functions $f : \mathbb{2} \to A$ such that $f(0) = x$ and $f(1) = y$. Obviously, one way to internalize this in ordinary homotopy type theory is to define

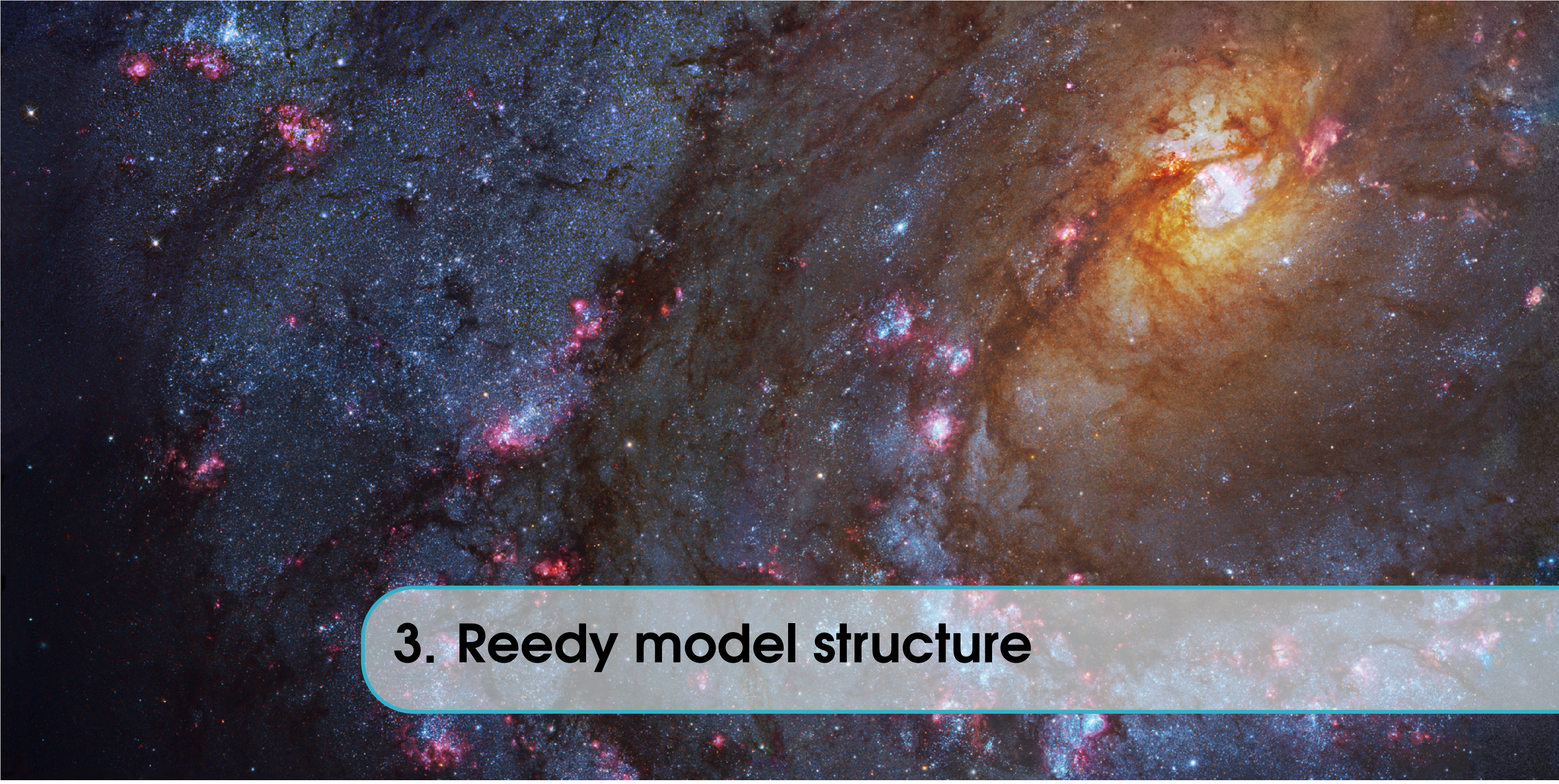
$$\hom_A(x,y) := \sum_{f:\mathbb{2}\to A} (x =_A f(0)) \times (f(1) =_A y).$$

But these equalities are then data, which have to be carried around everywhere. This is quite tedious, and the technicalities become nearly insurmountable when we come to define commutative triangles and commuatative squares. So, why not define $\hom_A(x,y)$ to be the type of functions $f : \mathbb{2} \to A$ such that $x \equiv f(0)$ and $f(1) \equiv y$? The first problem is that judgmental equality on A is interpreted by the diagonal $A \to A \times A$ in the slice category, which is usually not a fibration, unlike the path-object (see definition 3.1.8) $PA \to A \times A$, which interprets the identity type, is a fibration.

However, since $\mathbf{2} \to \mathbb{2}$ is a cofibration and $A \to \mathbf{1}$ is a fibration, we obtain that the pullback corner map $A^{\mathbb{2}} \to A \times A$ is a fibration (see lemma 3.4.1), which represents the the desired type family $\hom_A : A \times A \to \mathscr{U}$. There have been many approaches to internalize this argument, but we instead use a more refined approach due to Lumsdaine where we have a judgemental notion of a *cofibration*, and a new type former called an *extension type*: if $i : A \rightarrowtail B$ is a cofibration and $C : B \to \mathscr{U}$ is a type family with a section $d : \prod_{x:A} C(i(x))$, then there is a type $\left\langle \prod_{y:B} C(y)\big|_d^i \right\rangle$ of dependent functions $f : \prod_{y:B} C(y)$ such that $f(i(x)) \equiv d(x)$ for all $x : A$.

So now we have to give rules for what counts as a cofibration, in which we have to be careful to respect the semantics: it cannot simply be a map in any context that becomes a cofibration in the semantic slice category, since arbitrary slice categories are no longer cartesian monoidal model categories. However, we need not only $\mathbf{2} \to \mathbb{2}$ to be a cofibration, but also the inclusion of the boundary of any simplex $\partial\Delta^n \to \Delta^n$, and we would like these to be constructible in a sensible and uniform way rather than axiomatically asserted. One approach would be to keep the non-fibrant types with a notion of "strict pushout", and rules that cofibrations are closed under operations such as the "pushout product". We instead choose to keep all types fibrant, introducing rather a syntax for specifying cofibrations entirely separately from the rest of the type theory. Pleasingly, this separate syntax is exactly the coherent theory of a strict interval. We have a judgmental notion of a *shape*, representing the polytopes embedded in directed cubes that can be constructed in the theory of a strict interval, and we take the cofibrations to be the "inclusions of sub-shapes".

In the next chapter, we look at the general model structures in a category and particularly the Reedy model structure on bisimplicial sets.



3. Reedy model structure

3.1 Model structure on a category

A model structure on a category $\mathscr{C}$ is the data of three distinguished classes of morphisms which can be (loosely) thought of as follows:

- Weak equivalences, $W_{\mathscr{C}}$, which are the morphisms that we want to invert;
- Fibrations, $Fib_{\mathscr{C}}$, which play the role of surjections;
- Cofibrations, $Cof_{\mathscr{C}}$, which act like inclusions.

Definition 3.1.1 Given a commutative square of the following form:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

a lift in the diagram is a morphism $h : B \to X$ such that $hi = f$ and $ph = g$. A morphism $i : A \to B$ is said to have the left lifting property (LLP) with respect to another morphism $p : X \to Y$ and p is said to have the right lifting property (RLP) with respect to i if a lifting exists for any choice of f and g making diagram commute.

Definition 3.1.2 A morphism $f : A \to B$ in a category $\mathscr{C}$ is a retract of a morphism $g : C \to D$ in $\mathscr{C}$ if and only if there is a commutative diagram of the form:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B \end{tikzcd}$$

In particular f is a retract of g when viewed as objects in the arrow category.

Definition 3.1.3 A model category is a category $\mathscr{C}$ with three distinguished classes of morphisms:

- Weak equivalences—$W_{\mathscr{C}}$;
- Fibrations—$Fib_{\mathscr{C}}$;
- Cofibrations—$Cof_{\mathscr{C}}$;

each of which is closed under composition. We say that a morphism that is both a fibration and a weak equivalence is an acyclic fibration, and dually a morphism that is both a cofibration and a weak equivalence is an acyclic cofibration. The distinguished classes of morphisms and the category $\mathscr{C}$ must satisfy the following axioms:

MC1) $\mathscr{C}$ has all small limits and colimits. In particular, there is an initial object ϕ and a terminal object $*$.
MC2) If f and g are morphism such that gf is defined and if two of f , g, and gf are weak equivalences, then so is the third. That is, the weak equivalences satisfy the 2-out-of-3 property.
MC3) The three distinguished classes of morphisms are closed under retracts.
MC4) Given a commutative diagram of the form (2.1), a lift exists when either i is a cofibration and p is an acyclic fibration or when i is an acyclic cofibration and p is a fibration.
MC5) Each morphism f in $\mathscr{C}$ can be factored in two ways:
1. $f = pi$, where i is a cofibration and p is an acyclic fibration.
2. $f = pi$, where p is a fibration and i is an acyclic cofibration.

Definition 3.1.4 Let $\mathscr{C}$ be a model category. An object $X \in \mathscr{C}$ is said to be:

- Fibrant if the unique morphism $X \to *$ is a fibration;
- Cofibrant if the unique morphism $\phi \to X$ is a cofibration;
- Bifibrant if it is both fibrant and cofibrant.

■ **Example 3.1** Classical Quillen model structure on **Top**- weak equivalences are the weak homotopy equivalences. Fibrations are the Serre fibrations, maps which have the RLP with respect to all inclusions of the form $i : \mathbb{D}^n \to \mathbb{D}^n \times \mathbb{I}$ that include the n-disk as $\mathbb{D}^n \times \{0\}$. Cofibrations are the retracts of relative cell complexes. ■

■ **Example 3.2** Classical Quillen model structure on **sSet**- weak equivalences are weak homotopy equivalences, i.e morphisms whose geometric realization is a weak homotopy equivalence of topological spaces. Cofibrations are simply the monomorphisms $f : X \to Y$ which are precisely the levelwise injections, i.e morphisms of simplicial sets such that $f_n : X_n \to Y_n$ is an injection of sets for all $n \in \mathbb{N}$. Fibrations are the Kan fibrations, i.e maps $f : X \to Y$ which have the right lifting property with respect to all the horn inclusions:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

■

Definition 3.1.5 A weak factorization system on a category $\mathscr{C}$ is a pair $(L,R) \in \mathrm{Mor}(\mathscr{C}) \times \mathrm{Mor}(\mathscr{C})$ such that

1. Every morphism $f : X \to Y$ in $\mathscr{C}$ can be factorized as a composite $\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\in L"'] & & Y \\ & Z \arrow[ur, "\in R"'] & \end{tikzcd}$.

2. The classes are closed under having the lifting property with respect to one another. That is, L (resp., R) is the class of morphisms having the left (resp., right) lifting property against every morphisms in R (resp., L).

A model category give rise to two weak factorization systems $(\mathrm{Cof}_{\mathscr{C}}, \mathrm{Fib}_{\mathscr{C}} \cap W_{\mathscr{C}})$ and $(\mathrm{Cof}_{\mathscr{C}} \cap W_{\mathscr{C}}, \mathrm{Fib}_{\mathscr{C}})$.

Definition 3.1.6 Let $\mathscr{C}$ be a model category and $X \in \mathscr{C}$. A cylinder object $\mathrm{Cyl}(X)$ for X is a factorization of the codiagonal $\begin{tikzcd} X \sqcup X \arrow[rr, "\nabla_X"] \arrow[dr, "\in \mathrm{Cof}_{\mathscr{C}}"'] & & X \\ & \mathrm{Cyl}(X) \arrow[ur, "\in W_{\mathscr{C}}"'] & \end{tikzcd}$.

Definition 3.1.7 Let $f, g : X \to Y$ be a pair of morphisms in a model category. Then a left homotopy $\eta : f \sim_L g$ is a morphism $\eta : \mathrm{Cyl}(X) \to Y$ that makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d, "\eta"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

We also have a dual notion of path objects constructed out of fibration data.

Definition 3.1.8 Let $\mathscr{C}$ be a model category and $X \in C$. A path object $\mathrm{Path}(X)$ for X is a factorization of the diagonal $\begin{tikzcd} X \arrow[rr, "\Delta_X"] \arrow[dr, "\in W_{\mathscr{C}}"'] & & X \times X \\ & \mathrm{Path}(X) \arrow[ur, "\in \mathrm{Fib}_{\mathscr{C}}"'] & \end{tikzcd}$.

Definition 3.1.9 Let $f, g : X \to Y$ be a pair of morphisms in a model category. Then a right homotopy $\eta : f \sim_R g$ is a morphism $\eta : X \to \mathrm{Path}(Y)$ that makes the following diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "\eta"] \arrow[dr, "g"] & \\ Y & \mathrm{Path}(X) \arrow[l, "d_0"] \arrow[r, "d_1"'] & Y \end{tikzcd}$$

Definition 3.1.10 1. A pair of morphisms $f, g : X \to Y$ in a model category are homotopic, written $f \sim g$ if they are both left and right homotopic.

2. A morphism $f : X \to Y$ in a model category is a homotopy equivalence if there is a morphism $h : Y \to X$ such that $hf \sim \mathrm{id}_X$ and $fh \sim \mathrm{id}_Y$.

3.2 Reedy model structure on Fun$(\mathscr{R}, \mathscr{M})$

In this section we learn about the Reedy model structure on the functor category $\mathrm{Fun}(\mathscr{R}, \mathscr{M})$, where $\mathscr{R}$ is a Reedy category and $\mathscr{M}$ is a model category.

Definition 3.2.1 A Reedy category is a category $\mathscr{R}$ equipped with two wide subcategories $\mathscr{R}_+$ and $\mathscr{R}_-$ and a total ordering on the objects of $\mathscr{R}$ by an an ordinal-valued degree function such that
- Every nonidentity morphism in $\mathscr{R}_+$ raises degree.
- Every nonidentity morphism in $\mathscr{R}_-$ lowers degree.
- Every morphism f in $\mathscr{R}$ factors uniquely as a map in $\mathscr{R}_-$ followed by a map in $\mathscr{R}_+$.

■ **Example 3.3** The Reedy category structure on Δ is given by:
- The degree function $d : \mathrm{Ob}(\Delta) \to \mathbb{N}$ defined by $[k] \mapsto k$.
- A map $[k] \to [n]$ is in Δ_+ precisely if it is injective.
- A map $[n] \to [k]$ is in Δ_- precisely if it is surjective.

And the Reedy category structure on Δ^{op} is defined by switching Δ_+ and Δ_-. ■

Definition 3.2.2 Let $\mathscr{R}$ be a Reedy category and $\mathscr{M}$ be a model category, then the functor category $\mathrm{Fun}(\mathscr{R}, \mathscr{M})$ has a model structure in which a map $A \to B$ is

- A weak equivalence iff $A_x \to B_x$ is a weak equivalence in $\mathscr{M}$ for all $x \in \mathscr{R}$.
- A cofibration iff the induced map $A_x \sqcup_{L_x A} L_x B \to B_x$ is a cofibration in $\mathscr{M}$ for all $x \in \mathscr{R}$
- A fibration iff the induced map $A_x \to B_x \times_{M_x B} M_x A$ is a fibration in $\mathscr{M}$ for all $x \in \mathscr{R}$.

where $L_x A := \operatorname*{colim}_{r \to x \in \mathscr{R}_+} A_r$ and $M_x B := \lim_{x \to r \in \mathscr{R}_-} B_r$ are latching and matching objects of A and B at x respectively.

3.3 Reedy fibrations of bisimplicial sets

The category sSet:= $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets embeds in two "orthogonal" ways into the category ssSet:= $\mathrm{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ of bisimplicial sets. We express a bismplicial set as a bismplicial space via the isomorphism $\mathrm{ssSet} \cong \mathrm{sSet}^{\Delta^{\mathrm{op}}}$. We regard $X_{m,n}$ as the set of n-simplices in the mth space of the simplicial space $X : \Delta^{\mathrm{op}} \to \mathrm{sSet}$.

To define these two embeddings we use the external product bifunctor

$$\mathrm{sSet} \times \mathrm{sSet} \xrightarrow{\square} \mathrm{ssSet} \qquad (A \square B)_{m,n} := A_m \times B_n \quad .$$

Note that $\Delta^m \times \Delta^n$ is the functor represented by $(m,n) \in \Delta \times \Delta$. In particular, ssSet is a closed cartesian monoidal category and using exponential notation for the internal hom in ssSet, we have

$$(Y^X)_{m,n} = \mathrm{ssSet}(X \times (\Delta^m \square \Delta^n), Y) \quad .$$

Definition 3.3.1 Fixing one variable to be the point Δ^0, we obtain embeddings

$$\mathrm{disc} : \mathrm{sSet} \xrightarrow{\_\square\Delta^0} \mathrm{ssSet} \qquad \mathrm{const} : \mathrm{sSet} \xrightarrow{\Delta^0\square\_} \mathrm{ssSet}$$

of simplicial sets as discrete and constant bisimplicial sets, respectively. The discrete simplicial spaces factors as

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[rr, "X_{(\_)} \times \Delta^0"] \arrow[dr, "X"'] & & \mathrm{sSet} \\ & \mathrm{Set} \arrow[ur, "(\_) \times \Delta^0"'] & \end{tikzcd}$$

and the constant simplicial spaces factors as

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[rr, "\{\star\} \times X"] \arrow[dr, "\{\star\}"'] & & \mathrm{sSet} \\ & \mathbb{1} \arrow[ur, "(\_) \times X"'] & \end{tikzcd}$$

The discrete embedding positions the data of a simplicial set in the "categorical" direction, while the constant embedding positions the data in the "spacial" direction.

Definition 3.3.2 Given a simplicial object X in a locally small category $\mathscr{M}$ and a simplicial set S, define the weighted limit $\{S, X\}$ to be an object in $\mathscr{M}$ equipped with an isomorphism

$$\mathrm{Hom}_{\mathscr{M}}(\,\_, \{S, X\}) \cong \mathrm{Hom}_{\mathrm{sSet}}(S, \mathrm{Hom}_{\mathscr{M}}(\,\_, X))$$

of functors $\mathscr{M}^{op} \to \mathrm{Set}$.

Now onwards, we take $\mathscr{M} = \mathrm{sSet}$, where $\mathrm{Hom}_{\mathrm{sSet}}(A,X) : \Delta^{\mathrm{op}} \to \mathrm{Set}$ is defined by, $\mathrm{Hom}_{\mathrm{sSet}}(A,X)([m]) = \mathrm{Hom}_{\mathrm{sSet}}(A, X([m],\ \_))$.

Given a natural transformation $A \square B \to X$, i.e., collection of set maps $\{A_{[m]} \times B_{[n]} \to X([m],[n])\}_{m,n \geqslant 0}$, we would like to produce a natural transformation $B \to \{A, X\}$ and vice-versa. But by definition 3.3.2, it is equivalent to produce a natural transformation $A \to \mathrm{Hom}_{\mathrm{sSet}}(B,X)$, i.e., a collection of maps $\{A_{[m]} \to \mathrm{Hom}_{\mathrm{sSet}}(B, X([m],\ \_))\}_{m \geqslant 0}$.
But we have, $\mathrm{Hom}_{\mathrm{Set}}(A_{[m]} \times B_{[n]}, X([m],[n])) \cong \mathrm{Hom}_{\mathrm{Set}}(A_{[m]}, \mathrm{Hom}_{\mathrm{Set}}(B_{[n]}, X([m],[n])))$. Thus we have the adjoint pair:

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "A\Box\_"] \arrow[r, phantom, "\perp"] & \mathrm{ssSet} \arrow[l, bend left, "\{A,\_\}"] \end{tikzcd}$$

In particular, as a consequnce of Yoneda lemma, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{sSet}}(S,\{\Delta^m,X\}) &\cong \mathrm{Hom}_{\mathrm{ssSet}}(\Delta^m,\mathrm{Hom}_{\mathrm{sSet}}(S,X)) \\ \mathrm{Hom}_{\mathrm{sSet}}(S,\{\Delta^m,X\}) &\cong \mathrm{Hom}_{\mathrm{sSet}}(S,X([m],\ \_)) \\ \{\Delta^m,X\} &\cong X([m],\ \_) \end{aligned}$$

the mth column of X.

Definition 3.3.3 A morphism $X \to Y$ of bisimplicial sets is a Reedy fibration if and only if for all $m \geqslant 0$ the induced map

$$\{\Delta^m,X\} \to \{\partial\Delta^m,X\} \times_{\{\partial\Delta^m,Y\}} \{\Delta^m,Y\}$$

on weighted limits is a Kan fibration in sSet.

In the bisimplicial sets model, a dependent type family $C : A \to \mathscr{U}$ is modeled by a Reedy fibration $C \twoheadrightarrow A$.

Definition 3.3.4 A bisimplicial set X is Reedy fibrant just when the unique map $X \to 1$ is a Reedy fibration, which is the case when

$$\{\Delta^m,X\} \to \{\partial\Delta^m,X\}$$

is a Kan fibration in sSet.

In the bisimplicial sets model, a type is modeled by a Reedy fibrant bisimplicial set.

Also the bifunctor $\Box$ has an associated pushout product, that defines a biclosed bifunctor

$$\mathrm{sSet}^2 \times \mathrm{sSet}^2 \xrightarrow{\widehat{\Box}} \mathrm{ssSet}^2 \qquad .$$

The set of maps $\{(\partial\Delta^m \hookrightarrow \Delta^m)\widehat{\Box}(\partial\Delta^n \hookrightarrow \Delta^n)\}_{m,n\geqslant 0}$ defines a set of generating Reedy cofibrations for ssSet. A map of bisimplicial sets is a Reedy trivial fibration if and only if it has the right lifting property with respect to this set of maps.

3.4 Pullback power axiom

The Reedy fibrations enjoy the following important “Leibniz closure” property.

Lemma 3.4.1 If $i : U \to V$ is a cofibration and $p : X \twoheadrightarrow Y$ is a Reedy fibration then the map

$$\langle X^i, p^V\rangle : X^V \to X^U \times_{Y^U} Y^V,$$

which we denote by $\widehat{\{i,p\}}$, is a Reedy fibration, whose domain and codomain are Reedy fibrant if X and Y are, and which is a weak equivalence if p is.

Proof. The key here is to use the equivalence of pullback power axiom and the pushout product axiom that holds in a closed monoidal category. We show that if $i : U \to V$ and $j : A \to B$ are cofibrations of bisimplicial sets, then the pushout product map $i \widehat{\times} j$ is a cofibration that is trivial if j is.

$$\begin{tikzcd}
U \times A \arrow[r, "i \times 1_A"] \arrow[d, "1_U \times j"'] & V \times A \arrow[d, "k"] \arrow[ddr, bend left, "1_V \times j"] & \\
U \times B \arrow[r] \arrow[drr, "i \times 1_B"'] & \star \arrow[dr, dashed, "i \widehat{\times} j"] & \\
 & & V \times B
\end{tikzcd}$$

All the solid arrows in this diagram are monomorphisms and the outer square is a pullback, thus so is the dashed arrow, being a "union of subobjects" of $V \times B$. If j is acyclic, then since products of simplicial sets preserve weak equivalences, so do products of bisimplicial sets. Hence $1_U \times j$ and $1_V \times j$ are weak equivalences. Thus the map $1_U \times j$ is an acyclic cofibration so its pushout, the map denoted k in the diagram, is again a weak equivalence. Thus, by the 2-out-of-3 (MC2, see definition 3.1.3) property, $i \widehat{\times} j$ is a weak equivalence as well. ∎

3.5 Segal spaces

Definition 3.5.1 A Reedy fibrant bisimplicial set X is a Segal space if and only if for all $m \geqslant 2$ and $0 < i < m$ the induced map,

$$\{\Delta^m, X\} \to \{\Lambda^m_i, X\}$$

on weighted limits is a trivial fibration in sSet.

Proposition 3.5.1 A Reedy fibrant bisimplicial set X is a Segal space if and only if the induced map

$$X^{\Delta^2 \square \Delta^0} \to X^{\Lambda^2_1 \square \Delta^0}$$

is a Reedy trivial fibration.

Proof. Transposing across the adjunction between the cartesian product and internal hom for bisimplicial sets, $X^{\Delta^2 \square \Delta^0} \to X^{\Lambda^2_1 \square \Delta^0}$ is a Reedy trivial fibration if and only if X has the right lifting property with respect to the set of maps,

$$\{((\partial\Delta^m \hookrightarrow \Delta^m)\widehat{\square}(\partial\Delta^n \hookrightarrow \Delta^n))\widehat{\times}(\Lambda^2_1 \square \Delta^0 \hookrightarrow \Delta^2 \square \Delta^0)\}_{m,n \geqslant 0}.$$

This set is isomorphic to the set

$$\{((\partial\Delta^m \hookrightarrow \Delta^m)\widehat{\times}(\Lambda_1^2 \hookrightarrow \Delta^2))\widehat{\square}(\partial\Delta^n \hookrightarrow \Delta^n)\}_{m,n\geqslant 0},$$

where the left-hand product is now the cartesian product on sSet. Transposing across the weighted limit adjunction, we see that $X^{\Delta^2\square\Delta^0} \to X^{\Lambda_1^2\square\Delta^0}$ is a Reedy trivial fibration if and only if the induced map on weighted limits

$$\{\Delta^m \times \Delta^2 \to X\} \to \{\Delta^m \times \Lambda_1^2 \bigcup_{\partial\Delta^m\times\Lambda_1^2} \partial\Delta^m \times \Delta^2, X\}$$

is a trivial fibration of simplicial sets. Finally by the following combinatorial lemma of Joyal, this precisely characterizes the Segal spaces, which we state without a proof. ■

Lemma 3.5.2 The following sets generate the same class of morphisms of simplicial sets under coproduct, pushout, retract, and sequential composition:

(1) The inner horn inclusions $\Lambda_i^m \hookrightarrow \Delta^m$, for $m \geqslant 2$, $0 < i < m$.

(2) The collection of all inclusions

$$\{\Delta^m \times \Lambda_1^2 \bigcup_{\partial\Delta^m\times\Lambda_1^2} \partial\Delta^m \times \Delta^2 \hookrightarrow \Delta^m \times \Delta^2\}_{m\geqslant 0}.$$

Proof. See [16], [Lur09, 2.3.2.1] ■

In the next chapter, we finally focus on our main goal, and the first step towards it is to develop a three-layered type theory.

4. General type theory with shapes

Formally sHoTT is very similar to the recent "cubical type theories" studied by Cohen, Coquand, Huber, Mörtberg and others, whose basic setup can also be regarded as an instance of ours, using the theory of a de Morgan algebra. The most substantial difference is that our interval $\mathbb{2}$ describes extra structure in an “orthogonal” direction to the native “homotopy theory” of homotopy type theory, whereas the cubical interval $\mathbb{I}$ is rather a different way of describing that exact same native homotopy theory. This is why cubical type theory also includes the cubical Kan operations as rules of type theory. The closest analogue of this in our theory is the category structure of a Segal type induced by the contractibility of its composition spaces.
Our type theory is basically a three layered type theory, with the first two layers being ordinary coherent first-order logic, in which we express the theory of a strict interval $\mathbb{2}$. The third layer is then a homotopy type theory over the first two layers. In this chapter we describe the formal apparatus of the type theory.

4.1 The first layer: the layer of cubes

The first layer is a coherent theory of types called **cubes** with finite products of cubes and an axiomatic cube $\mathbb{2}$ with no other data. Below are the formal rules of the cube layer.

$$\frac{}{\mathbf{1}\ \text{cube}} \qquad \frac{}{\mathbb{2}\ \text{cube}} \qquad \frac{I\ \text{cube} \quad J\ \text{cube}}{I \times J\ \text{cube}} \qquad \frac{(t : I) \in \Xi}{\Xi \vdash t : I} \qquad \frac{}{\Xi \vdash \star : I}$$

$$\frac{\Xi \vdash s : I \quad \Xi \vdash t : J}{\Xi \vdash \langle s, t \rangle : I \times J} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \vdash \pi_1(t) : I} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \vdash \pi_2(t) : J}$$

Here Ξ is a context of variables belonging to cubes, and $\mathbf{1}$ denotes the empty product.

4.2 The second layer: the layer of topes

The second layer is an intuitionistic logic over the layer of cubes. We refer to its types as **topes**, thinking of them as polytopes embedded in a cube context. Topes admit operations of finite conjunction and disjunction, but not negation, implication, or either quantifier. There is also a basic "equality tope", which we write with the symbol $\equiv$, since it will be visible to the third layer as a strict or judgmental equality. Below are the formal rules of the tope layer, where Φ is a list of topes.

$$\frac{\phi \in \Phi}{\Xi \mid \Phi \vdash \phi} \qquad \frac{}{\Xi \vdash \top \text{ tope}} \qquad \frac{}{\Xi \mid \Phi \vdash \top} \qquad \frac{}{\Xi \vdash \bot \text{ tope}} \qquad \frac{\Xi \mid \Phi \vdash \bot}{\Xi \mid \Phi \vdash \psi}$$

$$\frac{\Xi \vdash \phi \text{ tope} \quad \Xi \vdash \psi \text{ tope}}{\Xi \vdash (\phi \wedge \psi) \text{ tope}} \qquad \frac{\Xi \mid \Phi \vdash \phi \quad \Xi \mid \Phi \vdash \psi}{\Xi \mid \Phi \vdash \phi \wedge \psi} \qquad \frac{\Xi \mid \Phi \vdash \phi \wedge \psi}{\Xi \mid \Phi \vdash \phi}$$

$$\frac{\Xi \mid \Phi \vdash \phi \wedge \psi}{\Xi \mid \Phi \vdash \psi} \qquad \frac{\Xi \vdash \phi \text{ tope} \quad \Xi \vdash \psi \text{ tope}}{\Xi \vdash (\phi \vee \psi) \text{ tope}} \qquad \frac{\Xi \mid \Phi \vdash \phi}{\Xi \mid \Phi \vdash \phi \vee \psi} \qquad \frac{\Xi \mid \Phi \vdash \psi}{\Xi \mid \Phi \vdash \phi \vee \psi}$$

$$\frac{\Xi \mid \Phi, \phi \vdash \chi \quad \Xi \mid \phi, \psi \vdash \chi \quad \Xi \mid \Phi \vdash \phi \vee \psi}{\Xi \mid \Phi \vdash \chi} \qquad \frac{\Xi \vdash s : I \quad \Xi \vdash t : I}{\Xi \vdash (s \equiv t) \text{ tope}}$$

$$\frac{\Xi \vdash s : I}{\Xi \mid \Phi \vdash (s \equiv s)} \qquad \frac{\Xi \mid \Phi \vdash (s \equiv t)}{\Xi \mid \Phi \vdash (t \equiv s)} \qquad \frac{\Xi \mid \Phi \vdash (s \equiv t) \quad \Xi \mid \Phi \vdash (t \equiv v)}{\Xi \mid \Phi \vdash (s \equiv v)}$$

$$\frac{\Xi \mid \Phi \vdash (s \equiv t) \quad \Xi, x : I \vdash \psi \text{ tope} \quad \Xi \mid \Phi \vdash \psi[s/x]}{\Xi \mid \Phi \vdash \psi[t/x]} \qquad \frac{\Xi \vdash t : I}{\Xi \mid \Phi \vdash t \equiv \star}$$

$$\frac{\Xi \vdash s : I \quad \Xi \vdash t : J}{\Xi \mid \Phi \vdash \pi_1(\langle s, t \rangle) \equiv s} \qquad \frac{\Xi \vdash s : I \quad \Xi \vdash t : J}{\Xi \mid \Phi \vdash \pi_2(\langle s, t \rangle) \equiv t} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \mid \Phi \vdash t \equiv \langle \pi_1(t), \pi_2(t) \rangle}$$

Now we define what do we mean by **shapes** in our theory. Roughly we can think of shapes as polytopes embedded in directed cubes.

Definition 4.2.1 A shape is a cube together with a tope in the corresponding singleton context. We could formalize this with a judgment and introduction rule such as the following:

$$\frac{I \text{ cube} \quad t : I \vdash \phi \text{ tope}}{\{t : I \mid \phi\} \text{ shape}}$$

4.3 The third layer: the extension types along cofibrations

There is a third layer of types that has all the ordinary type formers of homotopy type theory and one additional type former, the extension type. All the usual type formers and rules leave the cube and tope contexts unchanged. We include $\sum$-types, $\prod$-types with judgmental η-conversion, coproduct types, identity types $x : A, y : A \vdash x = y$ type, a universe $\mathscr{U}$ and so on. We assume function extensionality, but we will not need any higher inductive types, nor the univalence axiom.

In addition, we have various rules that relate the first two layers to the third. Firstly, we state all the rules in such a way that the following substitution/cut rules are admissible:

$$\frac{\Xi \vdash t : I \quad \Xi, x : I \mid \Phi \mid \Gamma \vdash a : A}{\Xi \mid \Phi[t/x] \mid \Gamma[t/x] \vdash a[t/x] : A[t/x]} \qquad \frac{\Xi \mid \Phi \vdash \psi \quad \Xi \mid \Phi, \psi \mid \Gamma \vdash a : A}{\Xi \mid \Phi \mid \Gamma \vdash a : A}$$

along with the obvious rules like weakening and contraction for the cube and tope contexts. Secondly, we have rules ensuring that the type theory respects the "tope logic" in a strict judgmental way. The appropriate sort of respect for $\top$ and $\wedge$ is already ensured by the cut and weakening rules. But in the case of $\bot$ and $\vee$, we have to assert elimination and computation rules, as shown below. Note that the rules for $\vee$ say that $\phi \vee \psi$ is a strict pushout of ϕ and ψ under $\phi \wedge \psi$, as is always the case in a coherent category.

$$\frac{\Xi \mid \Phi \vdash \bot}{\Xi \mid \Phi \mid \Gamma \vdash \mathrm{rec}_\bot : A} \qquad \frac{\Xi \mid \Phi \vdash \bot \quad \Xi \mid \Phi \mid \Gamma \vdash a : A}{\Xi \mid \Phi \mid \Gamma \vdash a \equiv \mathrm{rec}_\bot}$$

$$\frac{\begin{array}{c}\Xi \mid \Phi \vdash \phi \vee \psi \qquad \Xi \mid \Phi \mid \Gamma \vdash A \text{ type} \\ \Xi \mid \Phi, \phi \mid \Gamma \vdash a_\phi : A \quad \Xi \mid \Phi, \psi \mid \Gamma \vdash a_\psi : A \quad \Xi \mid \Phi, \phi \wedge \psi \mid \Gamma \vdash a_\phi \equiv a_\psi\end{array}}{\Xi \mid \Phi \mid \Gamma \vdash \mathrm{rec}_\vee^{\phi,\psi}(a_\phi, a_\psi) : A}$$

$$\frac{\begin{array}{c}\Xi \mid \Phi \vdash \phi \vee \psi \qquad \Xi \mid \Phi \mid \Gamma \vdash A \text{ type} \\ \Xi \mid \Phi, \phi \mid \Gamma \vdash a_\phi : A \quad \Xi \mid \Phi, \psi \mid \Gamma \vdash a_\psi : A \quad \Xi \mid \Phi, \phi \wedge \psi \mid \Gamma \vdash a_\phi \equiv a_\psi\end{array}}{\Xi \mid \Phi, \phi \mid \Gamma \vdash \mathrm{rec}_\vee^{\phi,\psi}(a_\phi, a_\psi) \equiv a_\phi}$$

$$\frac{\begin{array}{c}\Xi \mid \Phi \vdash \phi \vee \psi \qquad \Xi \mid \Phi \mid \Gamma \vdash A \text{ type} \\ \Xi \mid \Phi, \phi \mid \Gamma \vdash a_\phi : A \quad \Xi \mid \Phi, \psi \mid \Gamma \vdash a_\psi : A \quad \Xi \mid \Phi, \phi \wedge \psi \mid \Gamma \vdash a_\phi \equiv a_\psi\end{array}}{\Xi \mid \Phi, \psi \mid \Gamma \vdash \mathrm{rec}_\vee^{\phi,\psi}(a_\phi, a_\psi) \equiv a_\psi}$$

$$\frac{\Xi \mid \Phi \vdash \phi \vee \psi \qquad \Xi \mid \Phi \mid \Gamma \vdash a : A}{\Xi \mid \Phi \mid \Gamma \vdash a \equiv \mathrm{rec}_\vee^{\phi,\psi}(a, a)}$$

We also require the following compatibility rule, saying that tope equality behaves like judgmental equality

$$\frac{\Xi \mid \Phi \vdash (s \equiv t) \qquad \Xi, x : I \mid \Phi \mid \Gamma \vdash a : A}{\Xi \mid \Phi \mid \Gamma[s/x] \vdash a[s/x] \equiv a[t/x]}$$

Finally, we come to the reason for introducing this whole three-layer theory: extension types along cofibrations. As our notion of "cofibration" we use a shape inclusion, i.e. a pair of shapes $\{t : I \mid \phi\}$ and $\{t : I \mid \psi\}$ in the same cube such that $t : I \mid \phi \vdash \psi$. We will sometimes abbreviate this as $\{t : I \mid \phi\} \subseteq \{t : I \mid \psi\}$. Below are the formal rules for the extension types.

$$\frac{\begin{array}{c}\{t : I \mid \phi\} \text{ shape} \quad \{t : I \mid \psi\} \text{ shape} \quad t : I \mid \phi \vdash \psi \\ \Xi \mid \Phi \vdash \Gamma \text{ ctx} \qquad \Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash A \text{ type} \qquad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A\end{array}}{\Xi \mid \Phi \mid \Gamma \vdash \left\langle \prod_{t:I|\psi} A\big|^{\phi}_{a} \right\rangle \text{ type}}$$

$$\frac{\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash b : A \qquad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash b \equiv a}{\Xi \mid \Phi \mid \Gamma \vdash \lambda t^{I|\psi}.b : \left\langle \prod_{t:I|\psi} A\big|^{\phi}_{a} \right\rangle}$$

$$\frac{\Xi \mid \Phi \mid \Gamma \vdash f : \left\langle \prod_{t:I|\psi} A\big|^{\phi}_{a} \right\rangle \qquad \Xi \vdash s : I \quad \Xi \mid \Phi \vdash \psi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash f(s) : A}$$

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \phi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash f(s) \equiv a[s/t]}$$

$$\frac{\Xi \vdash s : I \qquad \Xi \mid \Phi \vdash \psi[s/t]}{\Xi \mid \Phi \mid \Gamma \vdash (\lambda t^{I|\psi}.b)(s) \equiv b[s/t]}$$

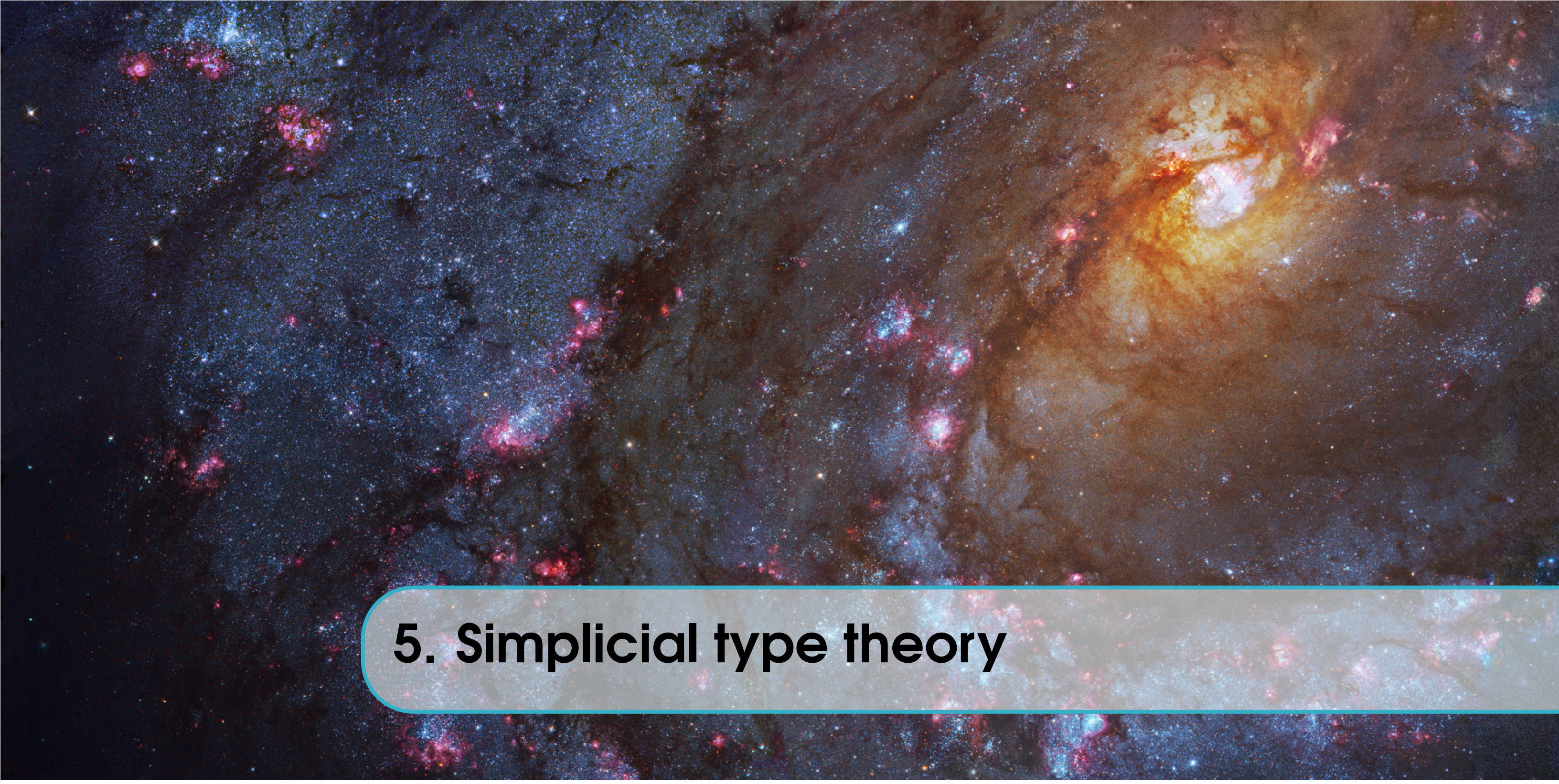
$$\frac{\Xi \mid \Phi \mid \Gamma \vdash f : \left\langle \prod_{t:I|\psi} A\big|^{\phi}_{a} \right\rangle}{\Xi \mid \Phi \mid \Gamma \vdash f \equiv \lambda t^{I|\psi}.f(t)}$$

In the formation rule, the judgement $\Xi \mid \Phi \vdash \Gamma$ ctx means that Γ is a well-formed context of types relative to $\Xi \mid \Phi$. The point is that Γ is not allowed to depend on t or ψ, and (implicitly) that Φ is also not allowed to depend on t. The type A, however, is allowed to depend on t and ψ, i.e. we allow "dependent extensions". Having just introduced extension types and their notation, we now proceed to introduce an abuse of that notation. The rules above are written in the usual formal type-theoretic way, with the dependent type A, tope Φ, and term $a : A$ being expression metavariables containing the variable $t : I$. Note that the variable t is bound in all three, i.e. its binding in $\prod_{t:I|\psi}$ scopes over the rest of the expression.

We can think of $\{t : I \mid \phi\}$ as a sub-shape of $\{t : I \mid \psi\}$ and read the judgment $\Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A$ as a function $\phi \to A$, we could represent a point in an extension type with a dashed arrow in the commutative diagram:

$$\begin{tikzcd} \phi \arrow[r, "a"] \arrow[d, hook] & A \\ \psi \arrow[ur, dashed] & \end{tikzcd}$$

In the next chapter, we will add the axioms of a strict interval to the formal apparatus of the the type theory we discuused in this chapter, that we will use in the rest of the chapters.



5. Simplicial type theory

In this chapter, we formulate the theory in which cube and tope layers form the coherent theory of a strict interval $\mathbb{2}$. Then we will define the most important shapes of our use, the simplices along with their boundaries and horns.

5.1 The strict interval

We have the axiomatic cube $\mathbb{2}$ and terms $0 : \mathbb{2}$, $1 : \mathbb{2}$ and an axiomatic inequality tope, $x : \mathbb{2}, y : \mathbb{2} \vdash (x \leqslant y)$ tope. The following strict interval axioms ensures that $\leqslant$ is a total order on $\mathbb{2}$:

$$
\begin{aligned}
x : \mathbb{2} \mid \cdot &\vdash (x \leqslant x) \\
x : \mathbb{2}, y : \mathbb{2}, z : \mathbb{2} \mid (x \leqslant y), (y \leqslant z) &\vdash (x \leqslant z) \\
x : \mathbb{2}, y : \mathbb{2} \mid (x \leqslant y), (y \leqslant x) &\vdash (x \equiv y) \\
x : \mathbb{2}, y : \mathbb{2} \mid \cdot &\vdash (x \leqslant y) \vee (y \leqslant x) \\
x : \mathbb{2} \mid \cdot &\vdash (0 \leqslant x) \\
x : \mathbb{2} \mid \cdot &\vdash (x \leqslant 1) \\
\cdot \mid (0 \equiv 1) &\vdash \bot
\end{aligned}
$$

5.2 Simplices and their subshapes

The interval type $\mathbb{2}$ allows us to define the simplices as the following shapes:

$$\Delta^n := \{\langle t_1, \ldots, t_n \rangle : \mathbb{2}^n \mid t_n \leqslant \ldots \leqslant t_1\}$$

This is an abuse of notation, since formally it should be written something like

$$\{t : (\ldots((\mathbb{2} \times \mathbb{2}) \times \mathbb{2})\ldots) \mid \pi_2(t) \leqslant \pi_2(\pi_1(t)) \leqslant \ldots \leqslant (\pi_1)^n(t)\}$$

but this will cause no harm . This is a meta-theoretic definition, as there is internally no "natural numbers" that we can use to parametrize a "family of shapes". Here is a list of few shapes of our interest:

$$
\begin{aligned}
\Delta^0 &:= \{t : 1 \mid \top\} \\
\Delta^1 &:= \{t : 2 \mid \top\} \\
\Delta^2 &:= \{\langle t_1, t_2 \rangle : 2 \times 2 \mid t_2 \leqslant t_1\} \\
\Delta^3 &:= \{\langle t_1, t_2, t_3 \rangle : 2 \times 2 \times 2 \mid t_3 \leqslant t_2 \leqslant t_1\} \\
\partial\Delta^1 &:= \{t : 2 \mid (t \equiv 0) \vee (t \equiv 1)\} \\
\partial\Delta^2 &:= \{(t_1, t_2) : \Delta^2 \mid (0 \equiv t_2 \leqslant t_1) \vee (t_1 \equiv t_2) \vee (t_2 \leqslant t_1 \equiv 1)\} \\
\Lambda^2_1 &:= \{\langle t_1, t_2 \rangle : 2 \times 2 \mid (t_1 = 1) \vee (t_2 = 0)\}
\end{aligned}
$$

One think to note is that the reversal of the order in the coordinates in the tope is just a matter of choice, so that i^{th} arrow is parametrized by t_i in the spine of a simplex. For instance, in a 3-simplex $f : \Delta^3 \to A$ with the following boundary:

$$
\begin{tikzcd}
 & \cdot \arrow[dd, "f_2"] \arrow[dr, "f_{23}"] & \\
\cdot \arrow[ur, "f_1"] \arrow[dr, "f_{12}"'] & & \cdot \\
 & \cdot \arrow[ur, "f_3"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \cdot \arrow[dr, "f_{23}"] & \\
\cdot \arrow[ur, "f_1"] \arrow[rr, "f_{123}"] \arrow[dr, "f_{12}"'] & & \cdot \\
 & \cdot \arrow[ur, "f_3"'] &
\end{tikzcd}
$$

we have,

$$f_1(t) \equiv f(t,0,0) \qquad f_2(t) \equiv f(1,t,0) \qquad f_3(t) \equiv f(1,1,t)$$

The other three 1-simplices are given by

$$f_{12}(t) \equiv f(t,t,0) \qquad f_{23}(t) \equiv f(1,t,t) \qquad f_{123}(t) \equiv f(t,t,t)$$

The other face and degeneracy operations between simplices can be defined in analogous ways. For instance, the four 2-simplex faces of a 3-simplex are obtained by requiring $0 \equiv t_3$, $t_3 \equiv t_2$, $t_2 \equiv t_1$, and $t_1 \equiv 1$ respectively. These yield operations on extension types:

$$
\begin{aligned}
\lambda f.\lambda\langle t_1, t_2\rangle.f\langle t_1, t_2, 0\rangle &: (\Delta^3 \to A) \to (\Delta^2 \to A) \\
\lambda f.\lambda\langle t_1, t_2\rangle.f\langle t_1, t_2, t_2\rangle &: (\Delta^3 \to A) \to (\Delta^2 \to A) \\
\lambda f.\lambda\langle t_1, t_2\rangle.f\langle t_1, t_1, t_2\rangle &: (\Delta^3 \to A) \to (\Delta^2 \to A) \\
\lambda f.\lambda\langle t_1, t_2\rangle.f\langle 1, t_1, t_2\rangle &: (\Delta^3 \to A) \to (\Delta^2 \to A)
\end{aligned}
$$

We will also use various sub-shapes of the simplices, particularly their boundaries and horns. The elimination rules for tope disjunction in chapter 4 ensure that terms depending on such a boundary can be "glued together" from terms depending on lower-dimensional simplices in the expected way. For example, to define a term $a : A$ in context $\partial\Delta^1 := \{t : 2 \mid (t \equiv 0) \vee (t \equiv 1)\}$, it is necessary and sufficient to give a term $a_0 : A$ in context $t : 2 \mid t \equiv 0$ and a term $a_1 : A$ in context $t : 2 \mid t \equiv 1$, such that if

$(t \equiv 0) \wedge (t \equiv 1)$ holds, then $a_0 \equiv a_1$. $(t \equiv 0) \vee (t \equiv 1)$ exactly behaves like the pushout of $(t \equiv 0)$ and $(t \equiv 1)$ under $(t \equiv 0) \wedge (t \equiv 1)$. But the last requirement is vacuous, since $(t \equiv 0) \wedge (t \equiv 1) \equiv \perp$ so that in that context, $a_0 \equiv a_1 \equiv \mathrm{rec}_\perp$. Moreover, since tope equality acts like judgmental equality, assuming $t : \mathbb{2}$ and $t \equiv 0$ is equivalent to assuming nothing at all, and similarly for assuming $t \equiv 1$. Therefore, a term $a : A$ in context $\partial\Delta^1$ is equivalently two terms $a_0, a_1 : A$ in no shape context, so that $\partial\Delta^1$ behaves like $\mathbf{2}$, the boolean type $\mathbf{1{+}1}$. Similarly, a term $a : A$ in context $\partial\Delta^2$ is equivalently three terms $a_0, a_1, a_2 : A$ in context $t : 2$ such that $a_0[0/t] \equiv a_1[0/t]$ and $a_0[1/t] \equiv a_2[0/t]$ and $a1[1/t] \equiv a_2[1/t]$.

5.3 Connection squares

We observe that, $\Delta^1 \times \Delta^1 := \{t : \mathbb{2} \times \mathbb{2} \mid \top\}$ behaves like the pushout of two copies of Δ^2 along their common diagonal boundary $\Delta^1_1 := \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid t \equiv s\}$.

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, dashed] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
\cdot \arrow[d] \arrow[dr] & \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\quad \vee \quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[dr] & \cdot \arrow[d] \\
& \cdot
\end{tikzcd}
$$

For since we have $t : \mathbb{2}, s : \mathbb{2} \vdash (t \leqslant s) \vee (s \leqslant t)$, a term $a : A$ in context $\Delta^1 \times \Delta^1$ is equivalently a term $a_0 : A$ in context $t : \mathbb{2}, s : \mathbb{2} \mid (t \leqslant s)$, which is Δ^2 upto tupling and permutation of variables and a term $a_1 : A$ in context $t : \mathbb{2}, s : \mathbb{2} \mid (s \leqslant t)$, another copy of Δ^2, such that if $(t \leqslant s) \wedge (s \leqslant t)$ holds, then $a_0 \equiv a_1$. But $(t \leqslant s), (s \leqslant t) \vdash t \equiv s$, so this the context $(t \leqslant s) \wedge (s \leqslant t)$ is a copy of Δ^1, embedded into the two copies of Δ^2 as one of the boundary edges.

Proposition 5.3.1 For any $f : \mathbb{2} \to A$, we have squares $\vee_f$, $\wedge_f : \mathbb{2} \times \mathbb{2} \to A$ with the faces

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, "f"] & y \arrow[d, Rightarrow] \\
y \arrow[r, Rightarrow] & y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
x \arrow[r, Rightarrow] \arrow[d, Rightarrow, "f"'] \arrow[dr, "f"] & x \arrow[d, "f"] \\
x \arrow[r, "f"'] & y
\end{tikzcd}
$$

such that

$$
\begin{array}{ll}
\vee_f(0,s) \equiv f(s) & \wedge_f(0,s) \equiv f(0) \\
\vee_f(t,0) \equiv f(t) & \wedge_f(t,0) \equiv f(0) \\
\vee_f(1,s) \equiv f(1) & \wedge_f(0,s) \equiv f(s) \\
\vee_f(t,1) \equiv f(1) & \wedge_f(t,0) \equiv f(t) \\
\vee_f(t,t) \equiv f(t) & \wedge_f(t,t) \equiv f(t)
\end{array}
$$

Proof. We define

$$
\begin{aligned}
\vee_f(t,s) &:= \mathrm{rec}_\vee^{t \leqslant s, s \leqslant t}(f(s), f(t)) \\
\wedge_f(t,s) &:= \mathrm{rec}_\vee^{t \leqslant s, s \leqslant t}(f(t), f(s))
\end{aligned}
$$

In both the cases, if $t \leqslant s$ and $s \leqslant t$, then $t \equiv s$, so is $f(t) \equiv f(s)$, so the the maps are well defined. Geometrically, $\vee_f$ glues two copies of the degenerate 2-simplex $\lambda t.\lambda s.f(t)$ along their common 1-simplex face, while $\wedge_f$ similarly glues two copies of the other degenerate 2-simplex $\lambda t.\lambda s.f(s)$. ■

As a second application, we observe that, at least as far as maps out of it are concerned, we may suppose Δ^2 to be a retract of $\Delta^1 \times \Delta^1$.

Proposition 5.3.2 For any type A, the type $\Delta^2 \to A$ is a retract of $\Delta^1 \times \Delta^1 \to A$.

Proof. We define the retraction by $\lambda f.\lambda\langle t,s\rangle.f(t,s) : (\Delta^1 \times \Delta^1 \to A) \to (\Delta^2 \to A)$. Then we define the section by $\lambda f.\lambda\langle t,s\rangle.\mathrm{rec}_\vee^{t\leqslant s, s\leqslant t}(f(t,t), f(t,s)) : (\Delta^2 \to A) \to (\Delta^1 \times \Delta^1 \to A)$. Again we check that, if $t \leqslant s$ and $s \leqslant t$ holds, then $t \equiv s$, so is $f(t,t) \equiv f(t,s)$, hence the section is well defined. The composite of the sectio followed by the retraction is, $f \mapsto \lambda\langle t,s\rangle.\mathrm{rec}_\vee^{t\leqslant s, s\leqslant t}(f(t,t), f(t,s)) \mapsto \lambda\langle t,s\rangle.(\lambda\langle t,s\rangle.\mathrm{rec}_\vee^{t\leqslant s, s\leqslant t}(f(t,t), f(t,s)))(t,s) \equiv \lambda\langle t,s\rangle.f(t,s) \equiv f$ since if $\langle t,s\rangle : \Delta^2$ then $s \leqslant t$. So, the composition is judgmentally equal to the identity. ■

Similar arguments apply in higher dimensions. For instance, the 3-dimensional "prism" $\Delta^2 \times \Delta^1 \equiv \{\langle\langle t_1,t_2\rangle,t_3\rangle \mid t_2 \leqslant t_1\}$ can be written as the union of three 3-simplices

$$\Delta^3 = \{\langle\langle t_1,t_2\rangle,t_3\rangle \mid t_3 \leqslant t_2 \leqslant t_1\}$$
$$\Delta^3 = \{\langle\langle t_1,t_2\rangle,t_3\rangle \mid t_2 \leqslant t_3 \leqslant t_1\}$$
$$\Delta^3 = \{\langle\langle t_1,t_2\rangle,t_3\rangle \mid t_2 \leqslant t_1 \leqslant t_3\}$$

along their common boundary 2-simplices

$$\Delta^2 = \{\langle\langle t_1,t_2\rangle,t_3\rangle \mid t_3 \equiv t_2 \leqslant t_1\}$$
$$\Delta^2 = \{\langle\langle t_1,t_2\rangle,t_3\rangle \mid t_2 \leqslant t_3 \equiv t_1\}$$

Proposition 5.3.3 For any type A, the type $\Delta^3 \to A$ is a retract of $\Delta^2 \times \Delta^1 \to A$.

Proof. We define the retraction by evaluating on the "middle" 3-simplex of the prism,

$$\begin{tikzcd}
& \cdot \arrow[d] \arrow[ddr] & \\
\cdot \arrow[ur] \arrow[r] \arrow[drr] & \cdot \arrow[dr] & \\
& & \cdot
\end{tikzcd}$$

$$\lambda f.\lambda\langle t_1,t_2,t_3\rangle.f\langle\langle t_1,t_3\rangle,t_2\rangle : (\Delta^2 \times \Delta^1 \to A) \to (\Delta^3 \to A).$$

This is well-defined since in Δ^3 we have $t_3 \leqslant t_2 \leqslant t_1$, hence in particular $t_3 \leqslant t_1$.
We define the section informally as a case split by,

$$\lambda f.\lambda\langle\langle t_1,t_2\rangle,t_3\rangle.f(t_1,t_2,t_2),\ t_3 \leqslant t_2$$
$$\lambda f.\lambda\langle\langle t_1,t_2\rangle,t_3\rangle.f(t_1,t_3,t_2),\ t_2 \leqslant t_3 \leqslant t_1$$
$$\lambda f.\lambda\langle\langle t_1,t_2\rangle,t_3\rangle.f(t_1,t_1,t_2),\ t_1 \leqslant t_3$$

Here in all cases we have $t_2 \leqslant t_1$, so in each case the requirement is met for f to be defined. The agreement on the boundary 2-simplices, when $t_3 \equiv t_1$ or $t_3 \equiv t_2$, is also obvious, as is the fact that this is a section of the above retraction. ■

6. Equivalences involving extension types

In this chapter we prove a bunch of important and interesting equivalences involving extension types which are generalizations of standard facts about ordinary and dependent function types.

6.1 Commutation/Swapping of arguments

Theorem 6.1.1 If $t : I \mid \phi \vdash \psi$ and $X : \mathscr{U}$, while $Y : \{t : I \mid \psi\} \to X \to \mathscr{U}$ and $f : \prod_{t:I|\phi} \prod_{x:X} Y(t,x)$, then

$$\left\langle \prod_{t:I|\psi} \left(\prod_{x:X} Y(t,x) \right) \Big|^{\phi}_{f} \right\rangle \simeq \prod_{x:X} \left\langle \prod_{t:I|\psi} Y(t,x) \Big|^{\phi}_{\lambda t.f(t,x)} \right\rangle$$

Proof. From left to right, define $g \mapsto \lambda x.\lambda t.g(t,x)$ and from right to left, define $h \mapsto \lambda t.\lambda x.h(x,t)$. Now, we have $g(t) \equiv f(t)$ assuming ϕ, and since evaluation respects judgemental equalities, $g(t,x) \equiv f(t,x)$ assuming ϕ. On the reverse direction, we have $h(x,t) \equiv f(t,x)$ assuming ϕ, so by η-expansion, $\lambda t.\lambda x.h(x,t) \equiv \lambda t.\lambda x.f(t,x) \equiv f$ assuming ϕ. So, both the maps are well-defined. The composites in both directions, $g \mapsto \lambda x.\lambda t.g(t,x) \mapsto \lambda t.\lambda x.(\lambda x.\lambda t.g(t,x))(x,t) \equiv \lambda t.\lambda x.g(t,x) \equiv g$ and $h \mapsto \lambda t.\lambda x.h(x,t) \mapsto \lambda x.\lambda t.(\lambda t.\lambda x.h(x,t))(t,x) \mapsto \lambda x.\lambda t.h(x,t) \equiv h$ respectively are judgementally equal to the identity, by η-expansion. Therefore, the left and right hand sides are equivalent. ■

6.2 Currying and the pushout product

Theorem 6.2.1 If $t : I \mid \phi \vdash \psi$ and $s : J \mid \chi \vdash \zeta$, while $X : \{t : I \mid \psi\} \to \{s : J \mid \zeta\} \to \mathscr{U}$ and $f : \prod_{<t,s>:I\times J|(\phi\wedge\zeta)\vee(\psi\wedge\chi)} X(t,s)$, then $\left\langle \prod_{t:I|\psi} \left\langle \prod_{s:J|\zeta} X(t,s) \Big|^{\chi}_{\lambda s.f(t,s)} \right\rangle \Big|^{\phi}_{\lambda t.\lambda s.f<t,s>} \right\rangle$
$\simeq \left\langle \prod_{<t,s>:I\times J|\psi\wedge\zeta} X(t,s) \Big|^{(\phi\wedge\zeta)\vee(\psi\wedge\chi)}_{f} \right\rangle \simeq \left\langle \prod_{s:J|\zeta} \left\langle \prod_{t:I|\psi} X(t,s) \Big|^{\phi}_{\lambda s.f(t,s)} \right\rangle \Big|^{\chi}_{\lambda t.\lambda s.f<t,s>} \right\rangle.$

The equivalence of the sides to the middle in Theorem 6.2.1 is a version of *currying*.
The shape $\{\langle t,s\rangle : I \times J \mid (\phi \wedge \zeta) \vee (\psi \wedge \chi)\}$ may be called the pushout product of the two shape

inclusions $\{t : I \mid \phi\} \subseteq \{t : I \mid \psi\}$ and $\{s : J \mid \chi\} \subseteq \{s : J \mid \zeta\}$. The following pushout square makes the picture more clear:

$$\begin{tikzcd}
\psi\wedge\zeta & & \\
& (\phi\wedge\zeta)\vee(\psi\wedge\chi) \arrow[ul, dashed] & \phi\wedge\zeta \arrow[l] \arrow[ull, bend right] \\
& \psi\wedge\chi \arrow[u] \arrow[uul, bend left] & \phi\wedge\chi \arrow[l] \arrow[u]
\end{tikzcd}$$

The dashed arrow in the diagram is due to the universal property of the pushout square which implies, $\langle t,s\rangle : I\times J \mid (\phi\wedge\zeta)\vee(\psi\wedge\chi) \vdash \psi\wedge\zeta$. The dashed arrow may be called the pushout product map.

Proof. The well-formedness of the extension types are trivial. Now the equivalence between the left- and right-hand types is again just application and re-abstraction, while the equivalence of both to the middle type is ordinary currying. For the first with the second, we perform currying, that is, from left to right $\lambda t.\lambda s.f\langle t,s\rangle \mapsto \lambda\langle t,s\rangle.f\langle t,s\rangle$ and right to left $\lambda\langle t,s\rangle.f\langle t,s\rangle \mapsto \lambda t.\lambda s.f\langle t,s\rangle$. Similarly, the second and the third are equivalent. ■

6.3 Generalization of the type theoretic principle of choice

Theorem 6.3.1 If $t : I \mid \phi \vdash \psi$, while $X : \{t : I \mid \psi\} \to \mathscr{U}$ and $Y : \prod_{t:I\mid\psi}(X \to \mathscr{U})$, while $a : \prod_{t:I\mid\phi} X(t)$ and $b : \prod_{t:I\mid\phi} Y(t, x(t))$, then

$$\left\langle \prod_{t:I\mid\psi}\left(\sum_{x:X(t)} Y(t,x)\right)\Big|^{\phi}_{\lambda t.(a(t),b(t))}\right\rangle \simeq \sum_{f:\left\langle \prod_{t:I\mid\psi} X(t)\big|^{\phi}_{a}\right\rangle} \left\langle \prod_{t:I\mid\psi} Y(t,f(t))\big|^{\phi}_{b}\right\rangle.$$

Proof. From left to right, define $h \mapsto (\lambda t.\pi_1(h(t)), \lambda t.\pi_2(h(t)))$ and from right to left, define $(f,g) \mapsto \lambda t.(f(t),g(t))$. Now, we have $h(t) \equiv \lambda t.(a(t),b(t))$ assuming ϕ, hence $\pi_1(h(t) \equiv a(t)$ and $\pi_2(h(t) \equiv b(t)$ assuming ϕ. On the reverse direction, we have $f(t) \equiv a(t)$ and $g(t) \equiv b(t)$ assuming ϕ, so $\lambda t.(f(t),g(t)) \equiv \lambda t.(a(t),b(t))$ assuming ϕ. So, both the maps are well defined. The composites in both directions, $h \mapsto (\lambda t.\pi_1(h(t)), \lambda t.\pi_2(h(t))) \mapsto \lambda t.(\lambda t.\pi_1(h(t))(t), \lambda t.\pi_2(h(t))(t)) \mapsto \lambda t.(\pi_1(h(t)), \pi_2(h(t)) \equiv h$ and $(f,g) \mapsto \lambda t.(f(t),g(t)) \mapsto (\lambda t.\pi_1(\lambda t.(f(t),g(t))(t)), \lambda t.\pi_2(\lambda t.(f(t), g(t))(t))) \mapsto (\lambda t.\pi_1((f(t),g(t)), \lambda t.\pi_2((f(t),g(t)) \equiv (\lambda t.f(t), \lambda t.g(t)) \equiv (f,g)$ respectively are judgementally equal to the identity, by β-reduction and η-expansion. Therefore, the left and right hand sides are equivalent. ■

6.4 Composites of cofibrations

Theorem 6.4.1 Suppose $t : I \mid \phi \vdash \psi$ and $t : I \mid \psi \vdash \chi$, and that $X : \{t : I \mid \chi\} \to \mathscr{U}$ and $a : \prod_{t:I\mid\phi} X(t)$. Then,

$$\left\langle \prod_{t:I\mid\chi} X\big|^{\phi}_{a}\right\rangle \simeq \left(\sum_{f:\left\langle \prod_{t:I\mid\psi} X\big|^{\phi}_{a}\right\rangle} \left\langle \prod_{t:I\mid\chi} X\big|^{\psi}_{f}\right\rangle\right).$$

Proof. From left to right, define $h \mapsto (\lambda t.h(t), \lambda t.h(t))$ and from right to left, define $(f,g) \mapsto \lambda t.g(t)$. Now,whenever $t : I$ such that χ holds, we have $h(t) \equiv a(t)$ assuming ϕ, so whenever $t : I$ such that ψ holds, we have $\lambda t.h(t) \equiv \lambda t.a(t)$ assuming ϕ, since $t : I \mid \psi \vdash \chi$. This implies, that whenever $t : I$ such that χ holds, we have $\lambda t.h(t) \equiv \lambda t.h(t)$ assuming ψ. On the reverse direction, whenever $t : I$ such that χ holds, we have $g(t) \equiv f(t)$ assuming ψ and whenever $t : I$ such that ψ holds, we have $f(t) \equiv a(t)$ assuming ϕ, so whenever $t : I$ such that χ holds, we have $g(t) \equiv a(t)$ assuming ϕ. So, both the maps are well-defined. The composites in both directions, $h \mapsto (\lambda t.h(t), \lambda t.h(t)) \mapsto \lambda t.\lambda t.h(t)(t) \equiv \lambda t.h(t) \equiv h$ and $(f,g) \mapsto \lambda t.g(t) \mapsto (\lambda t.\lambda t.g(t)(t), \lambda t.\lambda t.g(t)(t)) \equiv (\lambda t.g(t), \lambda t.g(t)) \equiv (\lambda t.f(t), \lambda t.g(t)) \equiv (f,g)$ since whenever $t : I$ assuming ψ, we have $\lambda t.g(t) \equiv \lambda t.f(t) : \left\langle \prod_{t:I|\psi} X\Big|^{\phi}_{a} \right\rangle$, are judgementally equal to the identity, by β-reduction and η-expansion. Therefore, the left and right hand sides are equivalent. ■

6.5 Unions of cofibrations

Theorem 6.5.1 Suppose $t : I \vdash \phi$ tope and $t : I \vdash \psi$ tope, and that we have $X : \{t : I \mid \phi \vee \psi\} \to \mathscr{U}$ and $a : \prod_{t:I|\psi} X(t)$. Then,

$$\left\langle \prod_{t:I|\phi\vee\psi} X\Big|^{\psi}_{a} \right\rangle \simeq \left\langle \prod_{t:I|\phi} X\Big|^{\phi\wedge\psi}_{\lambda t.a(t)} \right\rangle.$$

Proof. From left to right, define $h \mapsto \lambda t.h(t)$ and from right to left, define $g \mapsto \lambda t.rec_{\vee}^{\phi,\psi}(g(t), a(t))$. Since, $\phi \wedge \psi \vdash \psi$, $a(t)$ is defined whenever $t : I$ such that $\phi \wedge \psi$ holds. Since, $\phi \wedge \psi \vdash \phi \vdash \phi \vee \psi$, $h(t)$ is defined whenever $t : I$ such that $\phi \wedge \psi$ holds. So, $h(t) \equiv a(t)$ assuming $\phi \wedge \psi$ holds. On the reverse direction, the map is well defined because $g(t) \equiv a(t)$ whenever $t : I$ assuming $\phi \wedge \psi$ holds. Now, whenever $t : I$ such that ψ holds, we have $\lambda t.rec_{\vee}^{\phi,\psi}(g(t), a(t))(t) \equiv a(t)$. So, both the maps are well defined. The composites in both directions, $h \mapsto \lambda t.h(t) \mapsto \lambda t.rec_{\vee}^{\phi,\psi}(h(t), a(t)) \equiv \lambda t.h(t) \equiv h$, since if ϕ holds then $\lambda t.rec_{\vee}^{\phi,\psi}(h(t), a(t))(t) \equiv h(t)$ and if ϕ holds, then $\lambda t.rec_{\vee}^{\phi,\psi}(h(t), a(t))(t) \equiv a(t) \equiv h(t)$ and $g \mapsto \lambda t.rec_{\vee}^{\phi,\psi}(g(t), a(t)) \equiv \lambda t.\lambda t.rec_{\vee}^{\phi,\psi}(g(t), a(t))(t) \equiv \lambda t.g(t) \equiv g$, are judgementally equal to the identity, by β-reduction and η-expansion. Therefore, the left and right hand sides are equivalent. ■

6.6 Relative function extensionality

In this section we will assume a function extensionality axiom for extension types with respect to the homotopical identity types, similar to what we assume in HoTT.

Axiom 1 Supposing $t : I \mid \phi \vdash \psi$ and that $A : \{t : I \mid \psi\} \to \mathscr{U}$ is such that each $A(t)$ is contractible, and moreover $a : \prod_{t:I|\phi} A(t)$, then $\left\langle \prod_{t:I|\psi} A(t)\Big|^{\phi}_{a} \right\rangle$ is contractible.

Now suppose given $A : \{t : I \mid \psi\} \to \mathscr{U}$ and $a : \prod_{t:I|\phi} A(t)$, and also $f, g : \left\langle \prod_{t:I|\psi} A(t)\Big|^{\phi}_{a} \right\rangle$. Thus whenever $t : I$ such that ψ holds, we can form the identity type $f(t) = g(t)$, and thereby the extension type $\left\langle \prod_{t:I|\psi} f(t) = g(t)\Big|^{\phi}_{\lambda t.\mathrm{refl}} \right\rangle$, since $a(t) \equiv f(t)$ and $a(t) \equiv g(t)$ assuming ϕ. Ofcourse, we have $\lambda t^{I|\psi}.\mathrm{refl} : \left\langle \prod_{t:I|\psi} f(t) = f(t)\Big|^{\phi}_{\lambda t.\mathrm{refl}} \right\rangle$, so by identity elimination or path induction, we obtain the canonical map

$$(f = g) \to \left\langle \prod_{t:I|\psi} f(t) = g(t)\Big|^{\phi}_{\lambda t.\mathrm{refl}} \right\rangle.$$

Proposition 6.6.1 Assuming Axiom 1, we have the following:
(i) The map above is an equivalence.
(ii) In particular, for any $f, g : \left\langle \prod_{t:I|\psi} A(t) \Big|_{a}^{\phi} \right\rangle$, if $\left\langle \prod_{t:I|\psi} f(t) = g(t) \Big|_{\lambda t.\mathrm{refl}}^{\phi} \right\rangle$ is inhabited, then so is $f = g$.

Proof. It suffices to prove that for each f the induced map on total spaces

$$\left(\sum_{g:\left\langle \prod_{t:I|\psi} A(t) \big|_{a}^{\phi} \right\rangle} (f = g) \right) \to \left(\sum_{g:\left\langle \prod_{t:I|\psi} A(t) \big|_{a}^{\phi} \right\rangle} \left\langle \prod_{t:I|\psi} f(t) = g(t) \Big|_{\lambda t.\mathrm{refl}}^{\phi} \right\rangle \right)$$

is an equivalence. But the domain is contractible being a based path space. Now all that remains is to prove that the codomain is also contractible. By theorem 6.3.1,

$$\left\langle \prod_{t:I|\psi} \left(\sum_{y:A(t)} (f(t) = y) \right) \Big|_{\lambda t.\mathrm{refl}}^{\phi} \right\rangle \simeq \left(\sum_{g:\left\langle \prod_{t:I|\psi} A(t) \big|_{a}^{\phi} \right\rangle} \left\langle \prod_{t:I|\psi} f(t) = g(t) \Big|_{\lambda t.\mathrm{refl}}^{\phi} \right\rangle \right) .$$

Now, since each $\sum_{y:A(t)} (f(t) = y))$ is contractible being a based path space, by axiom 1, we have that $\left\langle \prod_{t:I|\psi} (\sum_{y:A(t)} (f(t) = y)) \Big|_{\lambda t.\mathrm{refl}}^{\phi} \right\rangle$ is contractible. ■

Another important consequence of the relative function extensionality is the famous homotopy extension property.

Proposition 6.6.2 Let $t : I \mid \phi \vdash \psi$. Assuming the relative function extensionality, if we have $A : \{t : I \mid \psi\} \to \mathscr{U}$ and $b : \prod_{t:I|\psi} A(t)$, and moreover $a : \prod_{t:I|\phi} A(t)$ and $e : \prod_{t:I|\phi} a(t) = b(t)$, then we have $a' : \left\langle \prod_{t:I|\psi} A(t) \Big|_{a}^{\phi} \right\rangle$ and $e' : \left\langle \prod_{t:I|\psi} a'(t) = b(t) \Big|_{e}^{\phi} \right\rangle$.

Proof. The type $\sum_{y:A(t)} (y = b(t))$ is contractible being a based path space and by Axiom 1, the extension type $\left\langle \prod_{t:I|\psi} \left(\sum_{y:A(t)} (y = b(t)) \right) \Big|_{\lambda t.(a(t),e(t))}^{\phi} \right\rangle$ is contractible, hence inhabited. Now by theorem 6.3.1, the type $\sum_{f:\left\langle \prod_{t:I|\psi} A(t) \big|_{a}^{\phi} \right\rangle} \left\langle \prod_{t:I|\psi} f(t) = b(t) \Big|_{e}^{\phi} \right\rangle$ is contractible, hence inhabited. Therefore we obtain our desired $a' : \left\langle \prod_{t:I|\psi} A(t) \Big|_{a}^{\phi} \right\rangle$ and $e' : \left\langle \prod_{t:I|\psi} a'(t) = b(t) \Big|_{e}^{\phi} \right\rangle$. ■

Now we prove that Axiom 1 follows from 6.6.1(ii) and 6.6.2.

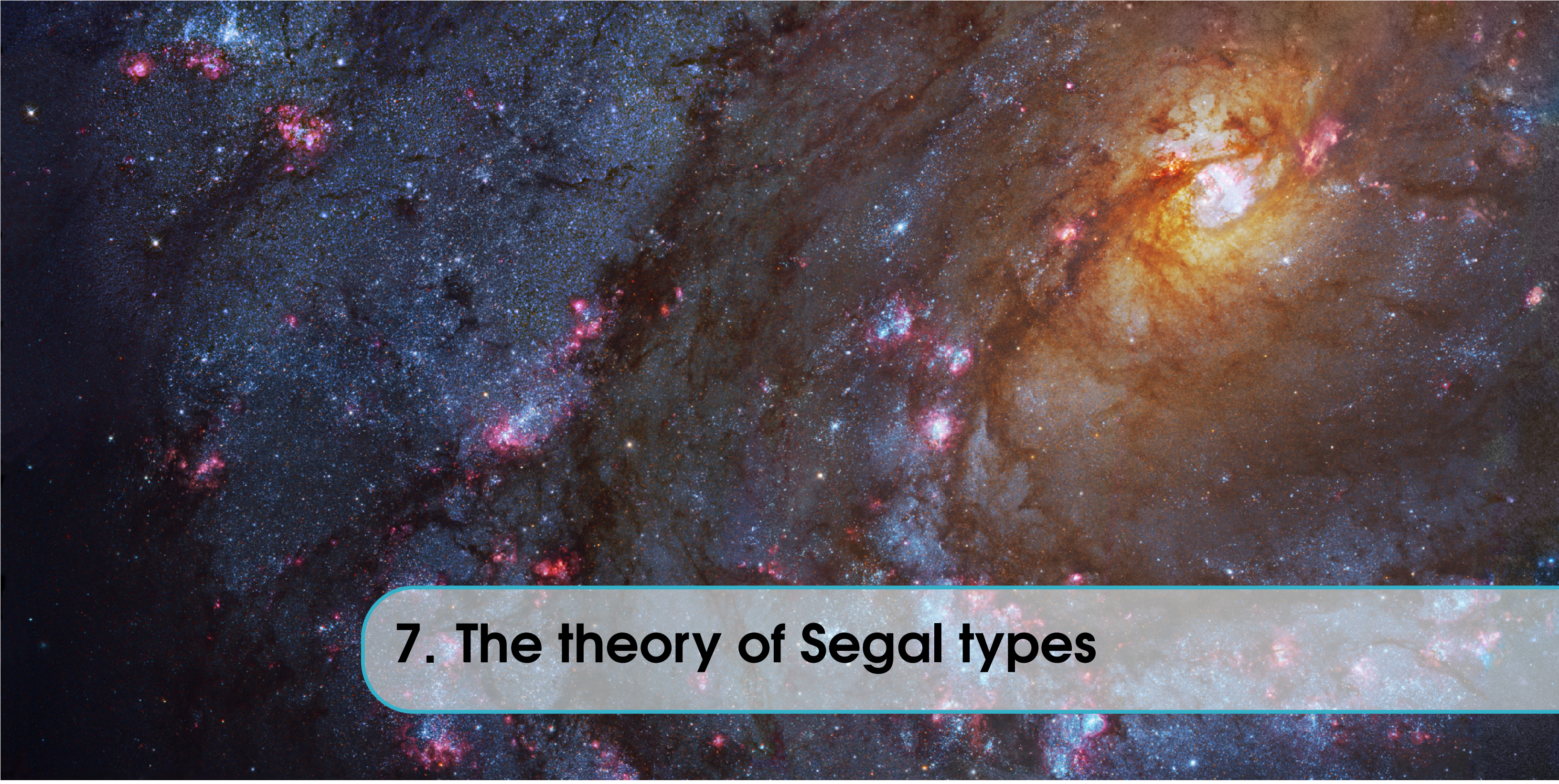
Proposition 6.6.3 If proposition 6.6.1(ii) and the homotopy extension property hold, then the relative function extensionality axiom holds.

Proof. Suppose $A : \{t : I \mid \psi\} \to \mathscr{U}$ and $a : \prod_{t:I|\phi} A(t)$ such that each $A(t)$ is contractible. The later assumption supplies centers of contraction $b(t)$ for each $t : I$ assuming ψ, hence we obtain a section $b : \prod_{t:I|\psi} A(t)$. Now, contractibility of each $A(t)$ also shows that if ϕ holds then, $a(t) = b(t)$ is inhabited, producing a section $e : \prod_{t:I|\phi} a(t) = b(t)$. Thus by the homotopy extension property, we have, $a' : \left\langle \prod_{t:I|\psi} A(t) \Big|_{a}^{\phi} \right\rangle$ and $e' : \left\langle \prod_{t:I|\psi} a'(t) = b(t) \Big|_{e}^{\phi} \right\rangle$.

Now let $f : \left\langle \prod_{t:I|\psi} A(t) \Big|_a^\phi \right\rangle$. Now since each $A(t)$ is contractible, we have $c : \prod_{t:I|\psi} f(t) = a'(t)$ and moreover, if ϕ holds, then $c(t) = \mathsf{refl}$ since any two paths in a contractible type are equal. Thus applying relative extensionality theorem to $f(t) = a'(t)$ in place of $A(t)$, c in place of b and $\lambda t.\mathsf{refl}$ in place of a, we obtain an inhabitant of $\left\langle \prod_{t:I|\psi} f(t) = a'(t) \Big|_{\lambda t.\mathsf{refl}}^\phi \right\rangle$ and by 6.6.1(ii), $f = a'$. ■

Proposition 6.6.4 Assuming Axiom 1, if $A : \{t : I \mid \psi\} \to \mathscr{U}$ and $a : \prod_{t:I|\phi} A(t)$ are such that each $A(t)$ is an n-type, then $\left\langle \prod_{t:I|\psi} A(t) \Big|_a^\phi \right\rangle$ is also an n-type.

Proof. When $n = -2$, it is exactly the Axiom 1. So, assume that if each $A(t)$ is a k-type, then $\left\langle \prod_{t:I|\psi} A(t) \Big|_a^\phi \right\rangle$ is also a k-type for some $k \geqslant -2$. Now let each $A(t)$ be a $(k+1)$-type and $f, g : \left\langle \prod_{t:I|\psi} A(t) \Big|_a^\phi \right\rangle$. In order to show that $f = g$ is a k-type, it suffices to show that $\left\langle \prod_{t:I|\psi} f(t) = g(t) \Big|_{\lambda t.\mathsf{refl}}^\phi \right\rangle$ is a k-type since assuming Axiom 1, $(f = g) \simeq \left\langle \prod_{t:I|\psi} f(t) = g(t) \Big|_{\lambda t.\mathsf{refl}}^\phi \right\rangle$ and equivalent types are equally truncated.
Now, since each $A(t)$ is a $(k+1)$-type, each $f(t) = g(t)$ is a k-type, so by induction hypothesis, $\left\langle \prod_{t:I|\psi} f(t) = g(t) \Big|_{\lambda t.\mathsf{refl}}^\phi \right\rangle$ is a k-type. Therefore it follows by induction that if each $A(t)$ is an n-type, then $\left\langle \prod_{t:I|\psi} a(t) \Big|_a^\phi \right\rangle$ is also an n-type. ■



7. The theory of Segal types

In this chapter, we use the simplices to parametrize the internal categorical structure in types satisfying an analogue of the famous Segal condition which we express in the internal language. We first define hom types of various dimensions whose terms are morphisms or compositions in another type.

Definition 7.0.1 Given $x, y : A$, determining a term $[x, y] : A$ in context $\partial\Delta^1$, we define,

$$\hom_A(x,y) := \left\langle \Delta^1 \to A \Big|^{\partial\Delta^1}_{[x,y]} \right\rangle .$$

We refer to an element of $\hom_A(x,y)$ as an arrow from x to y in A.

This plays the role of the directed hom-space of A. Note that every $f : \hom_A(x,y)$ is a kind of function from $\mathbb{2}$ to A, with the property that $f(0) \equiv x$ and $f(1) \equiv y$.

Definition 7.0.2 Given $x, y, z : A$ and $f : \hom_A(x,y)$, $g : \hom_A(y,z)$ and $h : \hom_A(x,z)$ we have an induced term $[x, y, z, f, g, h] : A$ in context $\partial\Delta^2$, and an extension type that we denote,

$$\hom_A^2\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \right) := \left\langle \Delta^2 \to A \Big|^{\partial\Delta^2}_{[x,y,z,f,g,h]} \right\rangle .$$

Definition 7.0.3 A **Segal type** is a type A such that for all $x, y, z : A$ and $f : \hom_A(x,y)$ and $g : \hom_A(y,z)$ the type,

$$\sum_{h:\hom_A(x,z)} \hom_A^2\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \right)$$

is contractible.

In particular, since the above type is contractible, it is inhabited. The first component of this inhabitant we call $g \circ f : \hom_A(x,z)$, the composite of g and f. The second component of this inhabitant is a 2-simplex in $\hom^2_A(f,g;g\circ f)$ which we think of as a "witness that $g\circ f$ is the composite of g and f", we denote it by $\mathrm{comp}_{g,f}$. The contractibility implies that composites are unique in the following sense: given $h:\hom_A(x,z)$ and any witness $p:\hom^2_A(f,g;h)$, we have $(h,p)=(g\circ f,\mathrm{comp}_{g,f})$, and hence in particular $h=g\circ f$.

Now we internalize the famous Segal condition in our type theory to characterize the Segal types.

Theorem 7.0.1 A type A is **Segal** if and only if the restriction map,

$$(\Delta^2\to A)\to(\Lambda^2_1\to A)$$

is an equivalence.

Proof. If Δ^1_1 denotes the diagonal 1-face $\{\langle s,t\rangle : \mathbb{2}\times\mathbb{2} \mid s=t\}$ of Δ^2, then we have $\Lambda^2_1\cap\Delta^1_1=\partial\Delta^1_1$ and $\Lambda^2_1\cup\Delta^1_1=\partial\Delta^2$. By theorem 6.5.1 to extend a map $\Lambda^2_1\to A$ to $\partial\Delta^2$ is equivalent to extending its restriction to $\partial\Delta^1_1$ to Δ^1_1. Now by theorem 6.4.1, we have

$$\sum_{h:\hom_A(x,z)} \hom^2_A\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z\end{tikzcd}\right) \equiv \sum_{h:\left\langle \Delta^1\to A\middle|^{\partial\Delta^1}_{[x,z]}\right\rangle} \left\langle \Delta^2\to A\middle|^{\partial\Delta^2}_{[x,y,z,f,g,h]}\right\rangle$$

$$\sum_{h:\left\langle \Delta^1\to A\middle|^{\partial\Delta^1}_{[x,z]}\right\rangle} \left\langle \Delta^2\to A\middle|^{\partial\Delta^2}_{[x,y,z,f,g,h]}\right\rangle \cong \sum_{l:\left\langle \partial\Delta^2\to A\middle|^{\Lambda^2_1}_{[x,y,z,f,g]}\right\rangle} \left\langle \Delta^2\to A\middle|^{\partial\Delta^2}_{l}\right\rangle$$

$$\sum_{l:\left\langle \partial\Delta^2\to A\middle|^{\Lambda^2_1}_{[x,y,z,f,g]}\right\rangle} \left\langle \Delta^2\to A\middle|^{\partial\Delta^2}_{l}\right\rangle \cong \left\langle \Delta^2\to A\middle|^{\Lambda^2_1}_{[x,y,z,f,g]}\right\rangle$$

In other words, $\sum_{h:\hom_A(x,z)} \hom^2_A\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z\end{tikzcd}\right)$ is the type of functions $\Delta^2\to A$ that restrict to f and g on Λ^2_1. Now, $\left\langle \Delta^2\to A\middle|^{\Lambda^2_1}_{[x,y,z,f,g]}\right\rangle$ is a fiber of the equivalence $(\Delta^2\to A)\to(\Lambda^2_1\to A)$, hence contractible. Therefore, by definition 7.0.3, A is Segal.
Conversely, using theorem 6.4.1 again, we have,

$$(\Delta^2\to A)\simeq\sum_{k:\Lambda^2_1\to A}\left\langle \Delta^2\to A\middle|^{\Lambda^2_1}_{k}\right\rangle.$$

Therefore, $\Delta^2\to A$ is the total space of a type family over $\Lambda^2_1\to A$ whose fibers are exactly the types $\sum_{h:\hom_A(x,z)} \hom^2_A\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z\end{tikzcd}\right)$. So, $(\Delta^2\to A)\to(\Lambda^2_1\to A)$ is a contractible map, hence an equivalence. ■

Corollary 7.0.2 If X is either a type or a shape and $A:X\to\mathscr{U}$ is such that each $A(x)$ is a Segal type for all $x:X$, then the dependent function type $\prod_{x:X}A(x)$ is a Segal type.

Proof. By theorem 6.1.1, we have $(\Delta^2 \to \prod_{x:X} A(x)) \simeq \prod_{x:X}(\Delta^2 \to A(x))$ and $(\Lambda_1^2 \to \prod_{x:X} A(x)) \simeq \prod_{x:X}(\Lambda_1^2 \to A(x))$. Now, the fiber of $\prod_{x:X}(\Delta^2 \to A(x)) \to \prod_{x:X}(\Lambda_1^2 \to A(x))$ at $g : \prod_{x:X}(\Lambda_1^2 \to A(x))$ is $\prod_{x:X}\left\langle \Delta^2 \to A(x)\Big|^{\Lambda_1^2}_{g(x)}\right\rangle$. Since $A(x)$ is Segal, $\left\langle \Delta^2 \to A(x)\Big|^{\Lambda_1^2}_{g(x)}\right\rangle$ is contractible, so is $\prod_{x:X}\left\langle \Delta^2 \to A(x)\Big|^{\Lambda_1^2}_{g(x)}\right\rangle$ by the relative function extensionality. Therefore, the map, $\prod_{x:X}(\Delta^2 \to A(x)) \to \prod_{x:X}(\Lambda_1^2 \to A(x))$ is contractible, hence an equivalence. So, $(\Delta^2 \to \prod_{x:X} A(x)) \simeq (\Lambda_1^2 \to \prod_{x:X} A(x))$, and by theorem 7.0.1, $\prod_{x:X} A(x)$ is a Segal type. ■

7.1 Identity

Identity morphisms in a Segal type are obtained as constant maps.

Definition 7.1.1 For any $x : A$, define a term $\mathbf{id}_x : \hom_A(x,x)$ by $\mathrm{id}_x(s) \equiv x$ for all $s : \mathbb{2}$.

Proposition 7.1.1 If A is a Segal type with terms $x, y : A$, then for any $f : \hom_A(x,y)$ we have $\mathrm{id}_y \circ f = f$ and $f \circ \mathrm{id}_x = f$.

Proof. For any $f : \hom_A(x,y)$ we have a canonical 2-simplex:

$$\lambda\langle s,t\rangle.f(s) : \left(\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd}\right)$$

to check that this indeed has the right boundary, we observe that $(s,0) \mapsto f(s)$ and $(s,s) \mapsto f(s)$, while $(1,t) \mapsto f(1) = y$. Thus, by the uniqueness of composites, $\mathrm{id}_y \circ f = f$.
Similarly, we also have a canonical 2-simplex:

$$\lambda\langle s,t\rangle.f(t) : \left(\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd}\right)$$

to check that this indeed has the left boundary, we observe that $(s,0) \mapsto f(0) = x$, while $(t,t) \mapsto f(t)$ and $(1,t) \mapsto f(t)$. Thus, by the uniqueness of composites, $f \circ \mathrm{id}_x = f$. ■

7.2 Associativity

We now prove that composition in a Segal type is associative.

Proposition 7.2.1 If A is Segal type with terms $x, y, z, w : A$, then for any $f : \hom_A(x,y)$, $g : \hom_A(y,z)$, $h : \hom_A(z,w)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Proof. By corollary 7.0.2, the type $A^{\mathbb{2}} := \mathbb{2} \to A$ is Segal. Thus, for any $f : \hom_A(x,y)$, $g : \hom_A(y,z)$, $h : \hom_A(z,w)$, the type

$$\sum_{p:\hom_{A^{\mathbb{2}}}(f,h)} \hom^2_{A^{\mathbb{2}}}\left(\begin{tikzcd} & y \arrow[dr, "g"] \arrow[dd] & \\ x \arrow[ur, "f"] \arrow[rr, "g"] \arrow[dd, "f"'] \arrow[ddrr, dashed, "m'"] & & z \arrow[dd, "h"] \\ & z \arrow[dr, "h"] & \\ y \arrow[ur, "g"] \arrow[rr] & & w \end{tikzcd}\right)$$

is contractible, hence inhabited. The second component of this inhabitant is a 2-simplex witness $\Delta^2 \times 2 \to A$, where the three rectangular faces of the triangular prism being $\mathrm{comp}_{g,f}$, $\mathrm{comp}_{h,g}$ and the first component of the witness. Now since, a rectangular prism is three 3-simplices glued together, we pick the middle shuffle $\lambda(t_1,t_2,t_3).((t_1,t_3),t_2) : \Delta^3 \to \Delta^2 \times 2$:

$$\begin{tikzcd}
 & y \arrow[d, "g"] \arrow[ddr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[r, "g\circ f"] \arrow[drr, "m'"'] & z \arrow[dr, "h"] & \\
 & & w
\end{tikzcd}$$

The front and the down faces are identified with further restrictions, $\lambda(s,t).(s,s,t) : \Delta^2 \to \Delta^3$ and $\lambda(s,t).(s,t,t) : \Delta^2 \to \Delta^3$ with a common edge $\lambda t.(t,t,t) : \Delta^1 \to \Delta^3$. This common edge is actually the inhabitant $m' : \hom_A(x,w)$. Now, since A is Segal, by the uniqueness of composites, $h \circ (g \circ f) = m'$ and also, $(h \circ g) \circ f = m'$. Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$. Basically what we are doing is we are extracting out the 3-simplex below out of the 2-simplex witness:

$$\begin{tikzcd}
 & y \arrow[dd, "g"] \arrow[dr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[dr, "g\circ f"'] \arrow[r, phantom, "\mathrm{comp}_{g,f}"] & & w \arrow[l, phantom, "\mathrm{comp}_{h,g}"] \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \arrow[dr, "h\circ g"] \arrow[d, phantom, "s"] & \\
x \arrow[ur, "f"] \arrow[rr, "m'"] \arrow[dr, "g\circ f"'] & {} \arrow[d, phantom, "r'"] & w \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}$$

■

7.3 Homotopies

Let A be a Segal type with terms $x, y : A$. Given two arrows $f, g : \hom_A(x,y)$, there are two ways to say that f and g are the same:

- we might have a path $p : f =_{\hom_A(x,y)} g$, or
- we might have a 2-simplex $q : \hom_A^2\left(\begin{tikzcd}[ampersand replacement=\&] \& x \arrow[dr, "f"] \& \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "g"'] \& \& y \end{tikzcd}\right)$

We now show that these two types are in fact equivalent:

Proposition 7.3.1 For any $f : \hom_A(x,y)$ and $g : \hom_A(y,z)$ and $h : \hom_A(x,z)$ in a Segal type A, the natural map

$$(g \circ f = h) \to \hom_A^2\left(\begin{tikzcd}[ampersand replacement=\&] \& y \arrow[dr, "g"] \& \\ x \arrow[ur, "f"] \arrow[rr, "h"'] \& \& y \end{tikzcd}\right)$$

is an equivalence.

Proof. The map is defined by path induction, since when $h \equiv g \circ f$ then, $\mathrm{comp}_{g,f} : \hom_A^2\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & y \end{tikzcd}\right)$.

To show that the map in the proposition is an equivalence, it suffices to show that the map of total spaces

$$\sum_{h:\hom_A(x,z)} (g \circ f = h) \to \sum_{h:\hom_A(x,z)} \hom_A^2\left(\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & y \end{tikzcd}\right)$$

is an equivalence. But both the total spaces are contractible, the left hand side being a based path space and the right hand side, since A is a Segal type. ■

Corollary 7.3.2 For $f, g : \hom_A(x,y)$ in a Segal type A, the natural map

$$(f = g) \to \hom_A^2\left(\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "g"'] & & y \end{tikzcd}\right)$$

is an equivalence.

Proof. Take $f = \mathrm{id}_x$, $g = f$, $h = g$, and then it follows from proposition 7.3.1. ■

Viewing homotopies as paths between arrows in a Segal type, behaves like a 2-category up to homotopy.

Proposition 7.3.3 Given $p : f =_{\hom_A(x,y)} g$ and $q : g =_{\hom_A(x,y)} h$ in a Segal type A, we have a concatenated equality $p \cdot q : f =_{\hom_A(x,y)} h$.

Proof. This is just a simple application of path induction, since we have a term $\mathrm{id}_{(x=z)} : (x = z) \to (x = z)$. ■

Proposition 7.3.4 Given $p : f =_{\hom_A(x,y)} g$ and $q : h =_{\hom_A(y,z)} k$ in a Segal type A, there is a concatenated equality $q \circ_2 p : h \circ f =_{\hom_A(x,z)} k \circ g$.

Proof. Another application of path induction, assuming $g \equiv f$ and $k \equiv h$, we have the terms $\mathrm{refl}_f : f =_{\hom_A(x,y)} f$ and $\mathrm{refl}_h : h =_{\hom_A(y,z)} h$. Now, define $\mathrm{refl}_h \circ_2 \mathrm{refl}_f := \mathrm{refl}_{h \circ f}$. ■

Proposition 7.3.5 Given $p : f =_{\hom_A(x,y)} g$ and $h : \hom_A(y,z)$ and $k : \hom_A(w,x)$ in a Segal type A, we have

$$\mathrm{refl}_h \circ_2 p = \mathrm{ap}_{(h\circ\_)}(p)$$
$$p \circ_2 \mathrm{refl}_k = \mathrm{ap}_{(\_\circ k)}(p).$$

Proof. Path induction on p. ■

Proposition 7.3.6 We have the following equality in a Segal type whenever it makes sense:

$$(q' \cdot p') \circ_2 (q \cdot p) = (q' \circ_2 q) \cdot (p' \circ_2 p).$$

Proof. Path induction on all four equalities. ■

On the other hand, if we view homotopies as 2-simplices, then a natural way to compose them is by filling 3-dimensional horns, as in a quasicategory. We can express this in terms of whiskering and concatenation of equalities.

Proposition 7.3.7 In a Segal type A, suppose given arrows f,g,h,k,l,m and equalities

$$p : g\circ f =_{\hom_A(x,z)} k \quad q : h\circ g =_{\hom_A(z,w)} l \quad r : h\circ k =_{\hom_A(x,w)} m$$

corresponding to 2-simplices that fill out the following horn $\Lambda^3_2 \to A$:

$$\begin{tikzcd}
 & y \arrow[dd, "g"] \arrow[dr, "l"] & \\
x \arrow[ur, "f"] \arrow[dr, "k"'] \arrow[r, phantom, "p"] & {} \arrow[r, phantom, "q"] & w \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \arrow[dr, "l"] \arrow[d, phantom, "-"] & \\
x \arrow[ur, "f"] \arrow[rr, "m"] \arrow[dr, "k"'] & {} \arrow[d, phantom, "r"] & w \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}$$

Then the horn has a filler $\Delta^3 \to A$ corresponding to the concatenated equality

$$l\circ f \overset{q}{=} (h\circ g)\circ f = h\circ(g\circ f) \overset{p}{=} h\circ k \overset{r}{=} m.$$

where p and q are whiskered by h and f respectively.

Proof. First step is to do path induction on p and q. This enables us to assume $k \equiv g\circ f$ and $l \equiv h\circ g$. This implies the 2-simplices corresponding to $p \equiv \mathrm{refl}$ and $q \equiv \mathrm{refl}$ are $\mathrm{comp}_{g,f}$ and $\mathrm{comp}_{h,g}$ respectively, while $l\circ f \overset{q}{=} (h\circ g)\circ f = h\circ(g\circ f) \overset{p}{=} h\circ k \overset{r}{=} m$ reduces to $(h\circ g)\circ f = h\circ(g\circ f) \overset{\mathrm{r}}{=} m$.

Next we recall that in proposition 7.2.1, we constructed a 3-simplex of the form

$$\begin{tikzcd}
 & y \arrow[dd, "g"] \arrow[dr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[dr, "g\circ f"'] \arrow[r, phantom, "\mathrm{comp}_{g,f}"] & {} \arrow[r, phantom, "\mathrm{comp}_{h,g}"] & w \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \arrow[dr, "h\circ g"] \arrow[d, phantom, "s"] & \\
x \arrow[ur, "f"] \arrow[rr, "m'"] \arrow[dr, "g\circ f"'] & {} \arrow[d, phantom, "r'"] & w \\
 & z \arrow[ur, "h"'] &
\end{tikzcd}$$

Now by the contractibility of the composition types for h and $g\circ f$, we have $(m',r') = (m,r)$. A crucial observation is that, type of 3-2-horns can be decomposed as

$$(\Lambda^3_2 \to A) \simeq \sum_{\alpha:\Delta^2\cup_{\Delta^1}\Delta^2\to A} \left\langle \Delta^2 \to A\Big|^{\Lambda^2_1}_{\alpha} \right\rangle$$

where $\Delta^2 \cup_{\Delta^1} \Delta^2$ denotes

$$\begin{tikzcd}
& y \arrow[dd, "g"] \arrow[dr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[dr, "g\circ f"'] \arrow[r, phantom, "\mathrm{comp}_{g,f}"] & {} \arrow[r, phantom, "\mathrm{comp}_{h,g}"] & w \\
& z \arrow[ur, "h"'] &
\end{tikzcd}$$

, with Λ^2_1 being the lower two 1-simplices

$$\begin{tikzcd}
x \arrow[dr, "g\circ f"'] & & w \\
& z \arrow[ur, "h"'] &
\end{tikzcd}$$

sitting inside it. Thus, equality $(m', r') = (m, r)$ in $\left\langle \Delta^2 \to A\Big|^{\Lambda^2_1}_{[h, g\circ f]} \right\rangle$ yields an equality of 3-2-horns [comp$_{g,f}$,comp$_{h,g}$,r'] = [comp$_{g,f}$,comp$_{h,g}$,r].
The 3-simplex

$$\begin{tikzcd}
& y \arrow[dd, "g"] \arrow[dr, "h\circ g"] & \\
x \arrow[ur, "f"] \arrow[dr, "g\circ f"'] \arrow[r, phantom, "\mathrm{comp}_{g,f}"] & {} \arrow[r, phantom, "\mathrm{comp}_{h,g}"] & w \\
& z \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& y \arrow[dr, "h\circ g"] \arrow[d, phantom, "s"] & \\
x \arrow[ur, "f"] \arrow[rr, "m'"'] \arrow[dr, "g\circ f"'] & {} \arrow[d, phantom, "r'"] & w \\
& z \arrow[ur, "h"'] &
\end{tikzcd}$$

is inhabitant of the type $\left\langle \Delta^3 \to A\Big|^{\Lambda^3_2}_{[\mathrm{comp}_{g,f}, \mathrm{comp}_{h,g}, r']} \right\rangle$, so the trick is to transport this term across the equality to get a term of $\left\langle \Delta^3 \to A\Big|^{\Lambda^3_2}_{[\mathrm{comp}_{g,f}, \mathrm{comp}_{h,g}, r]} \right\rangle$, our desired 3-simplex.

Finally, by the naturality of path transport, the missing 2-simplex face is the transport of s along the equality $m' = m$, which is equal to the concatenation $m' \overset{\mathrm{r'}}{=} h \circ (g \circ f) \overset{\mathrm{r}}{=} m$ of the two equalities induced by r and r'. Thus, the equality corresponding to this face is $(h \circ g) \circ f \overset{\mathrm{s}}{=} m' \overset{\mathrm{r'}}{=} h \circ (g \circ f) \overset{\mathrm{r}}{=} m$. But the concatenation of the first two of these equalities was the definition of associativity $(h \circ g) \circ f = h \circ (g \circ f)$ in proposition 7.2.1, so this is equal to $(h \circ g) \circ f = h \circ (g \circ f) \overset{\mathrm{r}}{=} m$. ■

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