Enriched Combinatorial Model categories

Arghan Dutta

October 2023

1 Definitions

We fix an universe **X**.

Definition 1.1 (Tensored category). Let $\mathcal{V}$ be a symmetric monoidal **X**-category and $\mathcal{C}$ be a $\mathcal{V}$-enriched **X**-category. Then $\mathcal{C}$ is *tensored* over $\mathcal{V}$ if for every v in $\mathcal{V}$ and c in $\mathcal{C}$, there exists an object $v \otimes^{\mathcal{V}}_{\mathcal{C}} c$ in $\mathcal{C}$, together with a natural isomorphism in $\mathcal{V}$

$$\underline{\mathrm{Mor}}^{\mathcal{V}}_{\mathcal{C}}(v \otimes^{\mathcal{V}}_{\mathcal{C}} c, c') \cong \underline{\mathcal{V}}(v, \underline{\mathrm{Mor}}^{\mathcal{V}}_{\mathcal{C}}(c, c'))$$

Definition 1.2 (Cotensored category). Let $\mathcal{V}$ be a symmetric monoidal **X**-category and $\mathcal{C}$ be a $\mathcal{V}$-enriched **X**-category. Then $\mathcal{C}$ is *cotensored* over $\mathcal{V}$ if for every v in $\mathcal{V}$ and c in $\mathcal{C}$, there exists an object $\mathrm{mor}^{\mathcal{V}}_{\mathcal{C}}(v, c)$ in $\mathcal{C}$, together with a natural isomorphism in $\mathcal{V}$

$$\underline{\mathrm{Mor}}^{\mathcal{V}}_{\mathcal{C}}(c', \mathrm{mor}^{\mathcal{V}}_{\mathcal{C}}(v, c)) \cong \underline{\mathcal{V}}(v, \underline{\mathrm{Mor}}^{\mathcal{V}}_{\mathcal{C}}(c', c))$$

Definition 1.3 (Quillen 2-variable adjunction). Let $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ be **X**-categories. An *adjunction of two variables* or a *2-variable adjunction* is a triplet $(\otimes, \underline{\mathrm{Mor}}, \mathrm{mor})$ consisting of bifunctors

$$\otimes : \mathcal{D} \times \mathcal{E} \longrightarrow \mathcal{F} \quad \underline{\mathrm{Mor}} : \mathcal{E}^{\mathrm{op}} \times \mathcal{F} \longrightarrow \mathcal{D} \quad \mathrm{mor} : \mathcal{D}^{\mathrm{op}} \times \mathcal{F} \longrightarrow \mathcal{E}$$

together with natural isomorphisms

$$\mathcal{F}(d \otimes e, f) \cong \mathcal{D}(d, \underline{\mathrm{Mor}}(e, f)) \cong \mathcal{E}(e, \mathrm{mor}(d, f))$$

Now suppose $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ are model **X**-categories and $(\otimes, \underline{\mathrm{Mor}}, \mathrm{mor})$ be an adjunction of two variables. Then, $(\otimes, \underline{\mathrm{Mor}}, \mathrm{mor})$ is a *Quillen adjunction of two variables* or a *2-variable Quillen adjunction* if it satisfies the following *pushout-product axiom* :

Axiom 1.1 (Pushout-product axiom). For any pair of cofibrations $f : Q \to R$ in $\mathcal{D}$ and $g : S \to T$ in $\mathcal{E}$, then the pushout-product

$$f \square g : (Q \otimes T) \coprod_{Q \otimes S} (R \otimes S) \longrightarrow R \otimes T$$

is a cofibration in $\mathcal{F}$, which is trivial if f or g is.

Definition 1.4 (Symmetric monoidal model category)**.** A *symmetric monoidal model* **X***-category* is a symmetric monoidal closed **X**-category $(\mathbf{V}, \otimes_{\mathbf{V}}, \mathbf{1}_{\mathbf{V}}, \underline{\mathrm{Mor}}_{\mathbf{V}})$, equipped with a model structure such that the following axiom holds:

Axiom 1.2 (Quillen adjunction axiom)**.** The triplet $(\otimes_{\mathbf{V}}, \underline{\mathrm{Mor}}_{\mathbf{V}}, \underline{\mathrm{Mor}}_{\mathbf{V}})$ is a Quillen adjunction of two variables.

Axiom 1.3 (Unit axiom)**.** For any object A, the canonical morphism,

$$Q_{\mathbf{V}} \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} A \longrightarrow \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} A \longrightarrow A$$

is a weak equivalence for the functorial cofibrant replacement $Q\mathbf{1}_{\mathbf{V}} \longrightarrow \mathbf{1}_{\mathbf{V}}$.

Definition 1.5 (Enriched model category)**.** Let $\mathcal{V}$ be a symmetric monoidal model **X**-category. Then a *model $\mathcal{V}$-category* is a tensored and cotensored $\mathcal{V}$-category $(\mathcal{C}, \underline{\mathrm{Mor}}_{\mathcal{C}}^{\mathcal{V}}, \otimes_{\mathcal{C}}^{\mathcal{V}}, \mathrm{mor}_{\mathcal{C}}^{\mathcal{V}})$, equipped with a model structure on the underlying **X**-category $\mathcal{C}$ such that the following axioms hold:

Axiom 1.4 (Quillen adjunction axiom)**.** The triplet $(\otimes_{\mathcal{C}}^{\mathcal{V}}, \underline{\mathrm{Mor}}_{\mathcal{C}}^{\mathcal{V}}, \mathrm{mor}_{\mathcal{C}}^{\mathcal{V}})$ is a Quillen adjunction of two variables.

Axiom 1.5 (Unit axiom)**.** For any object X in $\mathcal{C}$, the canonical morphism

$$Q_{\mathbf{V}} \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} X \longrightarrow \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} X \longrightarrow X$$

is a weak equivalence for the functorial cofibrant replacement $Q\mathbf{1}_{\mathbf{V}} \longrightarrow \mathbf{1}_{\mathbf{V}}$.

Definition 1.6 (λ-presentable object)**.** Let λ be a **X**-small regular cardinal and $\mathcal{C}$ be a **X**-category. An object K in $\mathcal{C}$ is called *λ-compact* or *λ-presentable* if the corresponding corepresentable functor

$$\mathrm{Mor}_{\mathcal{C}}(K, \_) : \mathcal{C} \longrightarrow \mathbf{Set}$$

commutes with all λ-filtered or λ-directed colimits.

Definition 1.7 (λ-accessible and locally λ-presentable categories)**.** A **X**-category $\mathcal{C}$ is called *λ-accessible* if:

1. $\mathcal{C}$ has λ-directed (or, equivalently λ-filtered) colimits.
2. $\mathcal{C}$ has a set $\mathcal{A}$ of λ-presentable objects, such that every object in $\mathcal{C}$ is a λ-directed colimit of objects from $\mathcal{A}$.

A **X**-category $\mathcal{C}$ is called *locally λ-presentable* if:

1. $\mathcal{C}$ is cocomplete.

2. $\mathcal{C}$ has a $\mathbf{X}$-small set $\mathcal{A}$ of λ-presentable objects, such that every object in $\mathcal{C}$ is a λ-directed colimit of objects from $\mathcal{A}$.

2 $\mathcal{H}$-(co)local objects and $\mathcal{H}$-(co)local equivalences

Let $\mathcal{M}$ be a model $\mathbf{X}$-category and $\mathcal{H}$ be a class of morphisms in $\mathcal{M}$.

Definition 2.1 ($\mathcal{H}$-(co)local object)**.** An object X of $\mathcal{M}$ is *$\mathcal{H}$-(co)local* if X is (co)fibrant and for every morphism $f : A \to B$ in $\mathcal{H}$, the induced morphism of homotopy function complexes (respectively, $f_* : \mathrm{Map}(X, A) \to \mathrm{Map}(X, B)$) $f^* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence.

Definition 2.2 ($\mathcal{H}$-(co)local equivalence)**.** A morphism $g : A \to B$ in $\mathcal{M}$ is a *$\mathcal{H}$-(co)local equivalence* if for every $\mathcal{H}$-(co)local object X, the induced morphism of homotopy function complexes (respectively, $g_* : \mathrm{Map}(X, A) \to \mathrm{Map}(X, B)$) $g^* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence.

3 Smith's Recognition Theorem

Theorem 3.1 (Smith)**.** *Suppose $\mathcal{C}$ is a locally $\mathbf{X}$-presentable $\mathbf{X}$-category, $\mathcal{W}$ a subcategory of $\mathcal{C}^{\mathbf{1}}$, and I an $\mathbf{X}$-small set of morphisms of $\mathcal{C}$. Suppose they satisfy the criteria:*

1. *$\mathcal{W}$ is closed under retracts and satisfies the 2-out-of-3 axiom.*

2. *The set $inj(I)$ is contained in $\mathcal{W}$.*

3. *The intersection $cof(I) \cap \mathcal{W}$ is closed under pushouts and transfinite composition.*

4. *$\mathcal{W}$ satisfies the solution set condition at I.*

Then $\mathcal{C}$ is combinatorial model categeory with weak equivalences $\mathcal{W}$, cofibrations $cof(I)$, and fibrations $inj(cof(I) \cap \mathcal{W})$.

Lemma 3.2. *Let $J \subseteq cof(I) \cap \mathcal{W}$ be a $\mathbf{X}$-small set of morphisms in $\mathcal{C}$ such that for any commutative sqaure,*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

with i in I, w in $\mathcal{W}$, there exists j in J that factors it:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \star \arrow[r] \arrow[d, "j"] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \star \arrow[r] & \bullet \end{tikzcd}$$

Then any f in $\mathcal{W}$ can be factored as $f = h \circ g$, where g is in cell(J) and h is in inj(I).

Proof. The proof is similar to the ordinary transfinite small object argument, it is just that we want the interpolating morphisms to be morphisms in J instead of I. Let $f : X \to Y$ be a morphism in $\mathcal{W}$ and set $P_0 = X, h_0 = f$. Having defined P_λ and $h_\lambda : P_\lambda \to Y$, now for a successor ordinal, let S_λ be the set of all commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & P_\lambda \arrow[d, "h_\lambda"] \\ B \arrow[r] & Y \end{tikzcd}$$

with i in I. The *density* assumption on J ensures te existence of a factorization

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & A_s \arrow[r, "t_s"] \arrow[d, "j_s"'] & P_\lambda \arrow[d, "h_\lambda"] \\ \bullet \arrow[r] & B_s \arrow[r] & Y \end{tikzcd}$$

with j_s in J, for each square s in S_λ. Let $P_{\lambda+1}$ be the pushout

$$\begin{tikzcd} \coprod A_s \arrow[r] \arrow[d, "\coprod j_s"'] & P_\lambda \arrow[d] \\ \coprod B_s \arrow[r] & P_{\lambda+1} \end{tikzcd}$$

along the canonical morphism $\coprod A_s \to P_\lambda$. Let $h_{\lambda+1}$ be the canonical pushout corner map from $P_{\lambda+1}$ to Y.

$$\begin{tikzcd} \coprod A_s \arrow[r] \arrow[d, "\coprod j_s"'] & P_\lambda \arrow[d] \arrow[ddr] & \\ \coprod B_s \arrow[r] \arrow[drr] & P_{\lambda+1} \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

The connecting map $P_\lambda \to P_{\lambda+1}$ is a pushout of coproducts of morphisms in J, which implies that the connecting map is in cell(J).
Now, at a limit ordinal λ, we define $P_\lambda = \text{colim}_{\alpha<\lambda} P_\alpha$ and $h_\lambda : P_\lambda \to Y$ to be the morphism induced by $\{h_\alpha : P_\alpha \to Y \mid \alpha < \lambda\}$.
Let now κ be a regular cardinal greater than the rank of presentability of all the domains of morphisms in I. The required factorization of f is $X \xrightarrow{g} P_\kappa \xrightarrow{h} Y$,

where g is a transfinite composition of morphisms in cell(J), hence in cell(J). So it remains to show that h is in inj(I). Indeed consider any lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & P_\kappa \arrow[d, "h"] \\ B \arrow[r] & Y \end{tikzcd}$$

where i is in I. Since P_κ is κ-filtered, $A \xrightarrow{a} P_\kappa$ factors through P_λ for some $\lambda < \kappa$. We have the following commutative square s in S_λ

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "i"'] & P_\lambda \arrow[d] \\ & P_\kappa \arrow[d, "h"] \\ B \arrow[r] & Y \end{tikzcd}$$

The lift in the original problem is the bottom composite in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \star \arrow[r] \arrow[d, "j_s"] & \bigstar \arrow[r] \arrow[d, "\coprod j_s"] & P_\lambda \arrow[d] & \\ B \arrow[r] & \star \arrow[r] & \bigstar \arrow[r] & P_{\lambda+1} \arrow[r] & P_\kappa \end{tikzcd}$$

□

Corollary 3.2.1. *Under the assumptions of the previous lemma, cof(J) = cof(I) $\cap\, \mathcal{W}$.*

Proof. cof(J) is the saturation of J, i.e, cof(J) is the smallest *weakly saturated* set of morphisms containing J, cof(J) $\subseteq$ cof(I) $\cap\, \mathcal{W}$. Conversely, consider any f in cof(I) $\cap\, \mathcal{W}$, by the previous lemma, $f = h \circ g$, where g is in cell(J) and h is in inj(I). By the retract argument, f is in cof(J). □

Lemma 3.3. *There exists a set* **X***-small set J satisfying the properties in lemma 3.1.*

Proof. Consider the set of all morphisms from i in I to the *solution set* $\mathcal{W}_i$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & X \arrow[d, "w_i"] \\ \bullet \arrow[r] & Y \end{tikzcd}$$

form the pushout and the canonical pushout corner map c

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & X \arrow[d, "\widetilde{i}"'] \arrow[ddr, "w_i"] & \\ \bullet \arrow[r] \arrow[drr] & P \arrow[dr, "c"] & \\ & & Y \end{tikzcd}$$

By the transfinite small object argument, c can be factored as $P \xrightarrow{p} Q \xrightarrow{q} Y$, where p is in cell(I) and q is in inj(I). Set $j = p \circ \widetilde{i}$. Let J be the set of all such j, one for each morphism from i in I to $\mathcal{W}_i$. Indeed, $\widetilde{i}$ is in cell(I), p is in cell(I) implying j is in cell$(I) \subseteq$ cof(I). Now, since $w_i = q \circ j$ and q is in inj$(I) \subseteq \mathcal{W}$, by the 2-out-of-3 axiom, j is in $\mathcal{W}$. Finally for any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

with i in I and w in $\mathcal{W}$, we have the required factorization

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & X \arrow[r, equal] \arrow[d, "j"'] & X \arrow[r] \arrow[d, "w_i"] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & Q \arrow[r, "q"] & Y \arrow[r] & \bullet \end{tikzcd}$$

□

Proof. (*Theorem 3.1*) Follows from lemma 3.2 and lemma 3.3. □