Enriched Combinatorial Model categories

Arghan Dutta

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1 Definitions

We fix an universe X.

Definition 1.1 (Tensored category). Let \mathcal{V} be a symmetric monoidal **X**-category and \mathcal{C} be a \mathcal{V} -enriched **X**-category. Then \mathcal{C} is *tensored* over \mathcal{V} if for every v in \mathcal{V} and c in \mathcal{C} , there exists an object $v \otimes_{\mathcal{C}}^{\mathcal{V}} c$ in \mathcal{C} , together with a natural isomorphism in \mathcal{V}

$$\underline{\mathrm{Mor}}_{\mathcal{C}}^{\mathcal{V}}(v \otimes_{\mathcal{C}}^{\mathcal{V}} c, c') \cong \underline{\mathcal{V}}(v, \underline{\mathrm{Mor}}_{\mathcal{C}}^{\mathcal{V}}(c, c'))$$

Definition 1.2 (Cotensored category). Let $\mathcal V$ be a symmetric monoidal **X**-category and $\mathcal C$ be a $\mathcal V$ -enriched **X**-category. Then $\mathcal C$ is *cotensored* over $\mathcal V$ if for every v in $\mathcal V$ and c in $\mathcal C$, there exists an object $\operatorname{mor}_{\mathcal C}^{\mathcal V}(v,c)$ in $\mathcal C$, together with a natural isomorphism in $\mathcal V$

$$\underline{\mathsf{Mor}}^{\mathcal{V}}_{\mathcal{C}}(c',\mathsf{mor}^{\mathcal{V}}_{\mathcal{C}}(v,c)) \cong \underline{\mathcal{V}}(v,\underline{\mathsf{Mor}}^{\mathcal{V}}_{\mathcal{C}}(c',c))$$

Definition 1.3 (Quillen 2-variable adjunction). Let \mathcal{D} , \mathcal{E} and \mathcal{F} be **X**-categories. An *adjunction of two variables* or a 2-variable adjunction is a triplet (\otimes , $\underline{\mathsf{Mor}}$, mor) consisting of bifunctors

$$\otimes: \mathcal{D} \times \mathcal{E} \longrightarrow \mathcal{F} \quad \text{Mor}: \mathcal{E}^{\text{op}} \times \mathcal{F} \longrightarrow \mathcal{D} \quad \text{mor}: \mathcal{D}^{\text{op}} \times \mathcal{F} \longrightarrow \mathcal{E}$$

together with natural isomorphisms

$$\mathcal{F}(d \otimes e, f) \cong \mathcal{D}(d, \underline{Mor}(e, f)) \cong \mathcal{E}(e, mor(d, f))$$

Now suppose \mathcal{D}, \mathcal{E} and \mathcal{F} are model **X**-categories and $(\otimes, \underline{Mor}, mor)$ be an adjunction of two variables. Then, $(\otimes, \underline{Mor}, mor)$ is a *Quillen adjunction of two variables* or a 2-variable Quillen adjunction if it satisfies the following pushout-product axiom:

Axiom 1.1 (Pushout-product axiom). For any pair of cofibrations $f: Q \to R$ in \mathcal{D} and $g: S \to T$ in \mathcal{E} , then the pushout-product

$$f \Box g : (Q \otimes T) \coprod_{O \otimes S} (R \otimes S) \longrightarrow R \otimes T$$

is a cofibration in \mathcal{F} , which is trivial if f or g is.

Definition 1.4 (Symmetric monoidal model category). A *symmetric monoidal model* **X**-category is a symmetric monoidal closed **X**-category ($V, \otimes_V, 1_V, \underline{Mor}_V$), equipped with a model structure such that the following axiom holds:

Axiom 1.2 (Quillen adjunction axiom). The triplet $(\otimes_{\mathbf{V}}, \underline{\mathsf{Mor}}_{\mathbf{V}}, \underline{\mathsf{Mor}}_{\mathbf{V}})$ is a Quillen adjunction of two variables.

Axiom 1.3 (<u>Unit axiom</u>). For any object *A*, the canonical morphism,

$$Q_{\mathbf{V}}\mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} A \longrightarrow \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} A \longrightarrow A$$

is a weak equivalence for the functorial cofibrant replacement $Q1_V \longrightarrow 1_V$.

Definition 1.5 (Enriched model category). Let \mathcal{V} be a symmetric monoidal model **X**-category. Then a *model* \mathcal{V} -category is a tensored and cotensored \mathcal{V} -category (\mathcal{C} , $\underline{\mathsf{Mor}}_{\mathcal{C}}^{\mathcal{V}}$, $\otimes_{\mathcal{C}}^{\mathcal{V}}$, $\mathsf{mor}_{\mathcal{C}}^{\mathcal{V}}$), equipped with a model structure on the underlying **X**-category \mathcal{C} such that the following axioms hold:

Axiom 1.4 (Quillen adjunction axiom). The triplet $(\otimes_{\mathcal{C}}^{\mathcal{V}}, \underline{\mathsf{Mor}}_{\mathcal{C}}^{\mathcal{V}}, \mathsf{mor}_{\mathcal{C}}^{\mathcal{V}})$ is a Quillen adjunction of two variables.

Axiom 1.5 (Unit axiom). For any object X in C, the canonical morphism

$$Q_{\mathbf{V}}\mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} X \longrightarrow \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} X \longrightarrow X$$

is a weak equivalence for the functorial cofibrant replacement $Q1_V \longrightarrow 1_V$.

Definition 1.6 (λ -presentable object). Let λ be a **X**-small regular cardinal and \mathcal{C} be a **X**-category. An object K in \mathcal{C} is called λ -compact or λ -presentable if the corresponding corepresentable functor

$$Mor_{\mathcal{C}}(K,_): \mathcal{C} \longrightarrow \mathbf{Set}$$

commutes with all λ -filtered or λ -directed colimits.

Definition 1.7 (*λ*-accessible and locally *λ*-presentable categories). A **X**-category \mathcal{C} is called *λ*-accessible if:

- 1. C has λ -directed (or, equivalently λ -filtered) colimits.
- 2. *C* has a set *A* of *λ*-presentable objects, such that every object in *C* is a *λ*-directed colimit of objects from *A*.

A **X**-category C is called *locally* λ -presentable if:

- 1. C is cocomplete.
- 2. C has a **X**-small set A of λ -presentable objects, such that every object in C is a λ -directed colimit of objects from A.

2 \mathcal{H} -(co)local objects and \mathcal{H} -(co)local equivalences

Let \mathcal{M} be a model **X**-category and \mathcal{H} be a class of morphisms in \mathcal{M} .

Definition 2.1 (\mathcal{H} -(co)local object). An object X of \mathcal{M} is \mathcal{H} -(co)local if X is (co)fibrant and for every morphism $f:A\to B$ in \mathcal{H} , the induced morphism of homotopy function complexes (respectively, $f_*:\operatorname{Map}(X,A)\to\operatorname{Map}(X,B)$) $f^*:\operatorname{Map}(B,X)\to\operatorname{Map}(A,X)$ is a weak equivalence.

Definition 2.2 (\mathcal{H} -(co)local equivalence). A morphism $g: A \to B$ in \mathcal{M} is a \mathcal{H} -(co)local equivalence if for every \mathcal{H} -(co)local object X, the induced morphism of homotopy function complexes (respectively, $g_*: \operatorname{Map}(X,A) \to \operatorname{Map}(X,B)$) $g^*: \operatorname{Map}(B,X) \to \operatorname{Map}(A,X)$ is a weak equivalence.

3 Smith's Recognition Theorem

Theorem 3.1 (Smith). Suppose C is a locally X-presentable X-category, W a subcategory of C^1 , and I an X-small set of morphisms of C. Suppose they satisfy the criteria:

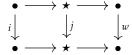
- 1. W is closed under retracts and satisfies the 2-out-of-3 axiom.
- 2. The set inj(I) is contained in W.
- 3. The intersection $cof(I) \cap W$ is closed under pushouts and transfinite composition.
- 4. W satisfies the solution set condition at I.

Then C is combinatorial model category with weak equivalences W, cofibrations cof(I), and fibrations $inj(cof(I) \cap W)$.

Lemma 3.2. Let $J \subseteq cof(I) \cap W$ be a **X**-small set of morphisms in C such that for any commutative square,



with i in I, w in W, there exists j in J that factors it:



Then any f in W can be factored as $f = h \circ g$, where g is in cell(J) and h is in inj(I).

Proof. The proof is similar to the ordinary transfinite small object argument, it is just that we want the interpolating morphisms to be morphisms in J instead of I. Let $f: X \to Y$ be a morphism in $\mathcal W$ and set $P_0 = X, h_0 = f$. Having defined P_λ and $h_\lambda: P_\lambda \to Y$, now for a successor ordinal, let S_λ be the set of all commutative squares

$$\begin{array}{ccc}
A & \longrightarrow & P_{\lambda} \\
\downarrow & & \downarrow h_{\lambda} \\
B & \longrightarrow & Y
\end{array}$$

with i in I. The *density* assumption on J ensures te existence of a factorization

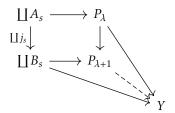
$$\bullet \longrightarrow A_{s} \xrightarrow{t_{s}} P_{\lambda}$$

$$\downarrow i \qquad \qquad \downarrow j_{s} \qquad \qquad \downarrow h_{\lambda}$$

$$\bullet \longrightarrow B_{s} \longrightarrow Y$$

with j_s in J, for each square s in S_{λ} . Let $P_{\lambda+1}$ be the pushout

along the canonical morphism $\coprod A_s \to P_{\lambda}$. Let $h_{\lambda+1}$ be the canonical pushout corner map from $P_{\lambda+1}$ to Y.



The connecting map $P_{\lambda} \to P_{\lambda+1}$ is a pushout of coproducts of morphisms in J, which implies that the connecting map is in cell(J).

Now, at a limit ordinal λ , we define $P_{\lambda} = \operatorname{colim}_{\alpha < \lambda} P_{\alpha}$ and $h_{\lambda} : P_{\lambda} \to Y$ to be the morphism induced by $\{h_{\alpha} : P_{\alpha} \to Y \mid \alpha < \lambda\}$.

Let now κ be a regular cardinal greater than the rank of presentability of all the domains of morphisms in I. The required factorization of f is $X \xrightarrow{g} P_{\kappa} \xrightarrow{h} Y$,

where g is a transfinite composition of morphisms in cell(J), hence in cell(J). So it remains to show that h is in inj(I). Indeed consider any lifting problem

$$\begin{array}{ccc}
A & \xrightarrow{a} & P_{\kappa} \\
\downarrow i & & \downarrow h \\
B & \longrightarrow & Y
\end{array}$$

where i is in I. Since P_{κ} is κ -filtered, $A \xrightarrow{a} P_{\kappa}$ factors through P_{λ} for some $\lambda < \kappa$. We have the following commutative square s in S_{λ}

$$\begin{array}{ccc}
A & \longrightarrow & P_{\lambda} \\
\downarrow & & \downarrow \\
i & & P_{\kappa} \\
\downarrow & & \downarrow h \\
B & \longrightarrow & Y
\end{array}$$

The lift in the original problem is the bottom composite in the diagram

Corollary 3.2.1. *Under the assumptions of the previous lemma,* $cof(J) = cof(I) \cap W$.

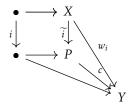
Proof. cof(J) is the saturation of J, i.e, cof(J) is the smallest *weakly saturated* set of morphisms containing J, $cof(J) \subseteq cof(I) \cap \mathcal{W}$. Conversely, consider any f in $cof(I) \cap \mathcal{W}$, by the previous lemma, $f = h \circ g$, where g is in cell(J) and h is in inj(I). By the retract argument, f is in cof(J).

Lemma 3.3. There exists a set **X**-small set J satisfying the properties in lemma 3.1.

Proof. Consider the set of all morphisms from i in I to the solution set W_i



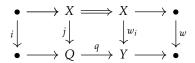
form the pushout and the canonical pushout corner map c



By the transfinite small object argument, c can be factored as $P \xrightarrow{p} Q \xrightarrow{q} Y$, where p is in $\operatorname{cell}(I)$ and q is in $\operatorname{inj}(I)$. Set $j = p \circ \widetilde{i}$. Let J be the set of all such j, one for each morphism from i in I to \mathcal{W}_i . Indeed, \widetilde{i} is in $\operatorname{cell}(I)$, p is in $\operatorname{cell}(I)$ implying j is in $\operatorname{cell}(I) \subseteq \operatorname{cof}(I)$. Now, since $w_i = q \circ j$ and q is in $\operatorname{inj}(I) \subseteq \mathcal{W}$, by the 2-out-of-3 axiom, j is in \mathcal{W} . Finally for any commutative square



with i in I and w in W, we have the required factorization



Proof. (*Theorem 3.1*) Follows from lemma 3.2 and lemma 3.3.

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